Math 240: Systems of Linear Differential Equations

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Outline

Today's Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Define systems of differential equations
2. Develop the notion of Linear Independence.
3. Develop the notion of General Solution.

An Example of a System of D.E.s

The dynamics of predictor and prey populations are modeled by the Lotka-Volterra equations

$$\frac{dx}{dt} = x(a - by)$$

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Where $x(t)$ is the population of prey at time t and $y(t)$ is the population of predators at time t.
This is a **non-linear** system

Linear systems

Definition

The following is a **first order system**

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + f_2(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + f_n(t)$$

Where each x_i is a function of t.

Examples of First Order Systems

Every n-th order linear differential equation can be written as an $n \times n$ first order system.

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Example Write $y'' - 3y' + 2y = 0$ as a first order system.

Solutions

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Given a system $X' = AX + F$ a **solution vector** is an $n \times 1$ column matrix with differential functions as entries that satisfies the system.

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The following is an **initial value problem** for a first order system $X' = AX + F$ and $X(t_0) = X_0$

Note: As long as everything in sight is continuous on an interval I containing t_0, then there exists a unique solution to the above IVP.

Supperposition Principle

Theorem

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Definition

Solution vectors $X_1, X_2, ..., X_k$ are **linearly independent** if

$$c_1X_1 + c_2X_2 + ... + c_nX_k = \mathbf{0}$$

implies $c_1 = c_2 = ... = c_n = 0.$

The Wronskian

Theorem

Let X_1, X_2, ..., X_n be n solution vectors to a homogeneous system on an interval I. They are linearly independent if and only if their **Wronskian** *is non-zero for every t in the interval.*

General Solutions to Homogeneous Systems

Theorem

Let $X_1, ..., X_n$ be a linearly independent set of solutions to a $n \times n$ first order **homogeneous** *linear system, then the general solution is*

$$X = c_1X_1 + c_2X_2 + ... + c_nX_n$$

where the c_i are arbitrary constants.

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Note: Assuming everything in sight is differentiable, general solutions always exist.

General Solutions to Non-homogeneous Systems

Theorem

Let X_p be a particular solution to a non-homogeneous first order linear system and X_h be the general solution to the associated homogeneous equation, then the **general solution** *is given by*

$$X = X_p + X_h$$