# Math 240: Cauchy-Euler Equations

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# Today's Goals

1. Learn how to solve Cauchy-Euler Equations.

# Cauchy-Euler Equations

**Goal:** To solve homogeneous DEs that are not constant-coefficient.

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**Definition**

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is a **Cauchy-Euler equation**.

# The 2nd Order Case

Try to solve

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

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If $m_1$ and $m_2$ are distinct real roots to $am(m-1) + bm + c = 0$, then the general solution to this DE is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

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# Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if $m$ is a root of multiplicity $k$, then

$$x^m, \ \ x^m ln(x), \ \ \ldots \ \ , x^m (ln(x))^{k-1}$$

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**Example:** What is the solution to

$$x^3 y''' + xy' - y = 0$$

# Conjugate Complex Roots

Given the DE

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + ...cy = 0$$

If $am(m-1) + bm + c = 0$ has complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, then the general solution is

$$y_g = x^\alpha[c_1 cos(\beta ln(x)) + c_2 sin(\beta ln(x))]$$

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**Example:** Solve $25x^2y'' + 25xy' + y = 0$