Math 240: Spring/Mass Systems II

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Monday, March 26, 2012

Outline

Today's Goals

1. Learn how to model spring/mass systems with damped motion.
2. Learn how to model spring/mass systems with driven motion.

Spring/Mass Systems with Damped Motion

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is now our model, where m is the mass, k is the spring constant, β is the damping constant and $x(t)$ is the position of the mass at time t.

Changing Variables

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and the roots of the Aux. Equation become

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2} \textit{ and } m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

Case 1: Overdamped

If $\lambda^2 - \omega^2 > 0$ the system is **overdamped** since β is large when compared to k. In this case the solution is

$$x = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{\sqrt{\lambda^2-\omega^2}t}).$$

Case 2: Critically Damped

If $\lambda^2 - \omega^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$x = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3: Underdamped

If $\lambda^2 - \omega^2 < 0$ the system is **underdamped** since k is large when compared to β. In this case the solution is

$$x = e^{-\lambda t}(c_1 cos(\sqrt{\omega^2 - \lambda^2} t) + c_2 sin(\sqrt{\omega^2 - \lambda^2} t))$$

.

Example

A 4 meter spring measures 8 meters long after a force of 16 newtons acts to it. A mass of 8 kilograms is attached to the spring. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 meters/sec.

Driven Motion

When an external force $f(t)$ acts on the mass on a spring, the equation for our model of motion becomes

$$\frac{d^2x}{dt^2} = -\frac{\beta}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{f(t)}{m}$$

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or in the language of λ and ω,

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$$

Example

When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest at equilibrium position. Starting at $t = 0$ a force of $f(t) = 65e^{-2t}$ is applied to the system. In the absence of damping, find the equation of motion.

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What is the amplitude of the oscillation after a very long time?

Transient and Steady State terms

Definition

In some cases, the solution to a D.E. can be written as the sum of a periodic function, $x_p(t)$, and a function that tends to zero as t tends to infinity, $x_c(t)$. In these cases, we say $x_p(t)$ is the **steady state term** and $x_c(t)$ is the **transient term**.