Math 240: Spring/Mass Systems I

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Outline

Today's Goals

1. Learn how to solve spring/mass systems.

The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... a_1 y' + a_0 y = g(x)$$

where $a_n, a_{n-1}, ..., a_0$ are constants.

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2. Step 2: Find a particular solution by guessing a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.

A Problem

Solve $y'' - 5y' + 4y = 8e^x$ using undetermined coefficients.

The solution

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by x^n, where n is the smallest positive integer that eliminates the duplication.

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Hooke's Law: The spring exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation.

$$F = ks$$

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Assuming free motion, **Newton's Second Law** states

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg = -kx$$

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$x(t) = c_1 cos(\omega t) + c_2 sin(\omega t)$.
Example: A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.