Math 240: Constant Coefficient Linear Differential Equations

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Friday March 16, 2012

Outline

Today's Goals

1. Use auxiliary equations to solve constant coefficient linear homogeneous D.E.s

Review

1. If $y_1, y_2, ..., y_n$ are linearly independent solutions to a homogeneous n-th order linear D.E., then $c_1y_1 + c_2y_2 + ... + c_ny_n$ is the general solution.

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2. If y_h is the homogeneous solution and y_p is the particular solution to a non-homogeneous linear D.E., then $y_h + y_p$ is the general solution.

A Motivating Example

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In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.

Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots m_1 and m_2, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

.

Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root m_1, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

.

Magic!

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \ldots$$

Magic!

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$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \ldots$$

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$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots)$$

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$$= cos(\theta) + isin(\theta)$$

Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$$

.

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0,$$

the **Auxiliary Equation** is

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The Auxiliary Equation determines the general solution.

General Solution from the Auxiliary Equation

1. If m is a real root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.

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1. If m is a real root of the auxiliary equation of multiplicity k then $e^{mx}, xe^{mx}, x^2e^{mx}, \ldots, x^{k-1}e^{mx}$ are linearly independent solutions.
2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then $e^{\alpha x}cos(\beta x), xe^{\alpha x}cos(\beta x), \ldots, x^{k-1}e^{\alpha x}cos(\beta x)$ and $e^{\alpha x}sin(\beta x), xe^{\alpha x}sin(\beta x), \ldots, x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.