Math 240: Linear Differential Equations

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Monday March 12, 2012

Today's Goals

Understand the form of solutions to the following types of higher order, linear differential equations

1. Initial Value Problems
2. Boundary Value Problems
3. Homogeneous and Nonhomogeneous Equations.

Differential equations

Definition

A differential equation is any equation involving a function, its derivatives.

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Solve $y' = y$

A Few Famous Differential Equations

1. Einstein's field equation in general relativity
2. The Navier-Stokes equations in fluid dynamics
3. Verhulst equation - biological population growth
4. The Black-Scholes PDE - models financial markets

Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

$$Solve: \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

$$Solve: \;\; a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

an **nth-order initial value problem**(IVP) is the above equation together with the following constraint

$$Subject\;\; to: \;\; y(x_0) = y_0, \;\; y'(x_0) = y_1, \;\; \ldots \;\; , y^{(n-1)}(x_0) = y_{n-1}$$

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If $g(x) = 0$, then we say the differential equation is **homogeneous**.

Existence and Uniqueness

Theorem

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $g(x)$ be continuous on and interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.

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Example: Does the following IVP have a unique solution? If so, on what intervals?

$y''' + y'' - y' - y = 9$ with $y(2) = 0$, $y'(2) = 0$ and $y''(2) = 0$

Boundary Value Problem

Definition

For a linear differential equation, an **nth-order boundary value problem**(BVP) is

$$Solve: \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to n equations that specify the value of y and its derivatives at **different** points (called **boundary conditions**).

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Question: What are the possible boundary conditions for a second order linear D.E.

One, Many or No Solutions

A BVP may have one, ∞-many, or no solutions.

Example: $x'' + 16x = 0$ subject to $x(0) = 0$ and $x(\frac{\pi}{2}) = 0$

Using Linearity to Find More Solutions

Theorem

(The Superposition Principle) Let $y_1, y_2, ...y_k$ be solutions to a homogeneous nth-order differential equation on an interval I. Then any linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x)$$

is also a solution, where $c_1, c_2, ..., c_k$ are constants.

Linear Independence of Functions

Definition

A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is **linearly dependent** on an interval I is there exists constants $c_1, c_2, ..., c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0$$

for every x in the interval. A set of functions that is not linearly dependent is said to be **Linearly Independent**.

The Wronskian

Definition

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possess at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

Linearly Independent Solutions

Theorem

Let $y_1, y_2, ..., y_n$ be n solutions to a homogeneous linear nth-order differential equation on an interval I. The the set of solutions is **linearly independent** *on I if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval. If the solutions $y_1, y_2, ..., y_n$ are linearly independent they are said to be a* **fundamental set of solutions***.*

Note: There always exists a fundamental set of solutions to an nth-order linear homogeneous differential equation on an interval I.

General Solution

Theorem

Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions set of solutions to an nth-order linear homogeneous differential equation on an interval I. Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where the c_i are arbitrary constants.

General Solutions to Nonhomogeneous Linear D.E.s

Theorem

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation on an interval I. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p$$

where the c_i are arbitrary constants.

Superposition Principle for Nonhomogeneous Equations

Theorem

Suppose y_{p_i} denotes a particular solution to the differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

Where $i = 1, 2, ..., k$. Then $y_p = y_{p_1} + y_{p_2} + ... + y_{p_k}$ is a particular solution of

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y =$$

$$g_1(x) + g_2(x) + ... + g_k(x)$$