Math 240: Divergence Theorem

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Outline

Stokes' Theorem

Theorem

Let S be an **nice** *oriented surface bounded by a* **nice** *curve C. Let $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a* **nice** *vector field. If C is traversed in the positive direction and $\mathbf{T}$ is the unit tangent vector to C then*

$$\oint_C F \circ d\mathbf{r} = \oint_C (F \circ \mathbf{T}) ds = \int\int_S (curl(F) \circ \mathbf{n}) dS$$

where $\mathbf{n}$ is the unit normal to S in the direction of the orientation of S.

Review Question: Let $F =< y^2, 2z + x, 2y^2 >$. Find a plane $ax + by + cz = 0$ such that $\oint_C F \circ dr = 0$ for every smooth simple closed curve C in the plane.

Today's Goals

1. Understand how to use the Divergence Theorem.

Divergence Theorem

Theorem

Let D be a **nice** *region in 3-space with* **nice** *boundary S oriented outward. Let F be a* **nice** *vector field. Then*

$$\int\int_S (F \circ \mathbf{n}) dS = \int\int\int_D div(F) dV$$

where $\mathbf{n}$ *is the unit normal vector to S.*

Divergence Theorem

Theorem

Let D be a **nice** *region in 3-space with* **nice** *boundary S oriented outward. Let F be a* **nice** *vector field. Then*

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where **n** *is the unit normal vector to S.*

Example Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$ and $z = 1$.

Divergence Theorem

Theorem

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where $\mathbf{n}$ *is the unit normal vector to S.*

Example: Use the divergence theorem to evaluate $\int\int_S (\mathbf{F} \cdot \mathbf{n})dS$ where $\mathbf{F} = < x + y, z, z - x >$ and S is the boundary of the region between $z = 9 - x^2 - y^2$ and the xy-plane.

Divergence Theorem

Theorem

Let D be a **closed and bounded** *region in 3-space with a* **piecewise smooth** *boundary S that is oriented outward. Let $F(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field* **for which P, Q and R are continuous and have continuous first partial derivatives in a region of 3-space containing D.** *Then*
$\int\int_S (F \circ \mathbf{n})dS = \int\int\int_D div(F)dV$
where $\mathbf{n}$ is the unit normal vector to S.

Divergence Theorem

Theorem

Let D be a **closed and bounded** *region in 3-space with a* **piecewise smooth** *boundary S that is oriented outward. Let $F(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field* **for which P, Q and R are continuous and have continuous first partial derivatives in a region of 3-space containing D.** *Then*
$\int\int_S (F \circ \mathbf{n}) dS = \int\int\int_D div(F) dV$
where **n** *is the unit normal vector to S.*

Example Find the outward flux of

$$\frac{< x, y, z >}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

across the region D given by $0 < a^2 \leq x^2 + y^2 + z^2 \leq b^2$.