Math 240: Inverses and Eigenvalues

Ryan Blair

University of Pennsylvania

Friday February 24, 2012

Outline

Today's Goals

1. Know the properties of inverses.
2. Be able to solve systems of linear equations using matrices.
3. Know how to interpret matrices as maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.
4. Know how to find eigenvalues.
5. Know how to find eigenvectors.

Matrix Inverse

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

In this case, B is the **inverse** of A.

Properties of Inverses

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $det(A^{-1}) = \frac{1}{det(A)}$
6. A is invertible if and only if $det(A) \neq 0$
7. A is invertible if and only if A has maximal rank.

Solving a Linear System Using Inverses

Let A be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

$$x = A^{-1}B$$

Solving a Linear System Using Inverses

Let A be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

$$x = A^{-1}B$$

Example: Solve the following linear system using inverses.

$$x + z = -4$$

$$x + y + z = 0$$

$$5x - y = 6$$

General maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

The following is a general map from $\mathbb{R}^n$ with coordinates $x_1, x_2, ..., x_n$ to $\mathbb{R}^m$ with coordinates $w_1, w_2, ..., w_m$.

$w_1 = f_1(x_1, x_2, ..., x_n)$
$w_2 = f_2(x_1, x_2, ..., x_n)$
...
$w_m = f_m(x_1, x_2, ..., x_n)$

General maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

The following is a general map from $\mathbb{R}^n$ with coordinates $x_1, x_2, ..., x_n$ to $\mathbb{R}^m$ with coordinates $w_1, w_2, ..., w_m$.

$w_1 = f_1(x_1, x_2, ..., x_n)$
$w_2 = f_2(x_1, x_2, ..., x_n)$
...
$w_m = f_m(x_1, x_2, ..., x_n)$

Example: Rewrite $g(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$ in this form.

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

The following is a general linear map from $\mathbb{R}^n$ with coordinates $x_1, x_2, ..., x_n$ to $\mathbb{R}^m$ with coordinates $w_1, w_2, ..., w_m$.

$w_1 = a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n$

$w_2 = a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n$

...

$w_m = a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n$

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

The following is a general linear map from $\mathbb{R}^n$ with coordinates $x_1, x_2, ..., x_n$ to $\mathbb{R}^m$ with coordinates $w_1, w_2, ..., w_m$.

$w_1 = a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n$

$w_2 = a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n$

...

$w_m = a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n$

Key idea: Matrix-vector multiplication always encodes a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be encoded as Matrix-vector multiplication.

Types of Linear Maps

The following are types of linear maps

1. Reflection about a line in R^2
2. Reflection about a plane in R^3
3. Orthogonal projection onto an axis in R^2
4. Orthogonal projection onto a plane in R^3
5. Rotation about the origin in R^2

Types of Linear Maps

The following are types of linear maps

1. Reflection about a line in R^2
2. Reflection about a plane in R^3
3. Orthogonal projection onto an axis in R^2
4. Orthogonal projection onto a plane in R^3
5. Rotation about the origin in R^2

Types of Linear Maps

The following are types of linear maps

1. Reflection about a line in R^2
2. Reflection about a plane in R^3
3. Orthogonal projection onto an axis in R^2
4. Orthogonal projection onto a plane in R^3
5. Rotation about the origin in R^2

Key idea: Eigenvectors are vectors sent to scalar copies of themselves under the linear map corresponding to A.