Math 240: Inverses

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Outline

1. Properties of Determinants
2. Matrix Inverse
3. Properties of Inverses
4. Solving a Linear System Using Inverses

Using Elementary Row Operations to Find the Determinant

Suppose B is obtained from A by:

1. multiplying a row by a non-zero scalar c, then $det(A) = \frac{1}{c}det(B)$.
2. switching rows, then $det(A) = -det(B)$.
3. adding a multiple of one row to another row, then $det(A) = det(B)$.

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Using this idea we can quickly find determinants by row-reducing to triangular form.
Example: Find the determinant of the following matrix.

$$\begin{pmatrix} 0 & 2 & 0 & -3 \\ 3 & 0 & 2 & 5 \\ -2 & 4 & 0 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Properties of Determinants

Theorem

If elementary row or column operations lead to one of the following conditions, then the determinant is zero.

1. *an entire row (or column) consists of zeros.*
2. *one row (or column) is a multiple of another row (or column).*

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Let A and B be $n \times n$ matrices and c be a scalar.

1. $det(AB) = det(A)det(B)$
2. $det(cA) = c^n det(A)$
3. $det(A^T) = det(A)$

Today's Goals

1. Be able to find the inverse of a matrix or show it has no inverse.
2. Know the properties of inverses.
3. Be able to solve systems of linear equations using matrices.

Matrix Inverse

Definition

An $n \times n$ matrix A is **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

In this case, B is the **inverse** of A.

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3. If A is invertible, its inverse is denoted A^{-1}.

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Example: check the the following matrices are inverses of each other.

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix}$$

A 2×2 Matrix Inverse Formula

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix and $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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Exercise: Prove the above statement

Inverses of Arbitrary $n \times n$ Matrices

How to find the inverse of an arbitrary $n \times n$ matrix A.

1. Form the augmented $n \times 2n$ matrix $[A|I_n]$.
2. Find the reduced row echelon form of $[A|I_n]$.
3. If $rank(A) < n$ then A is not invertible.
4. If $rank(A) = n$, then the RREF form of the augmented matrix is $[I_n|A^{-1}]$.

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Find the inverse of $\begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

Properties of Inverses

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $det(A^{-1}) = \frac{1}{det(A)}$
6. A is invertible if and only if $det(A) \neq 0$

Solving a Linear System Using Inverses

Let A be invertible and $Ax = B$ be a linear system, then the solution to the linear system is given by

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Example: Solve the following linear system using inverses.

$$x + z = -4$$

$$x + y + z = 0$$

$$5x - y = 6$$