Math 240: Determinants

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Outline

Rank

Definition

Let A be an $m \times n$ matrix. The **rank** of A is the maximal number of linearly independent row vectors

Definition

(Pragmatic)
Let A be an $m \times n$ matrix and B be its row-echelon form. The **rank** of A is the number of pivots of B.

Today's Goals

1. Be able to find determinants using cofactor expansion.
2. Be able to find determinants using row operations.
3. Know the properties of determinants.

Determinant of a 2×2 Matrix

Definition

Give a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of A is

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minors and Cofactors

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Given a matrix A the **cofactor**, C_{ij}, is given by the following formula

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Definition of Arbitrary Determinant

Definition

Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix.
The cofactor expansion of A along the ith row is

$$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

The cofactor expansion of A along the jth column is

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Special Matrices and Determinants

Definition

An $n \times n$ matrix $A = (a_{ij})_{n\times n}$ is **lower triangular** if $a_{ij} = 0$ whenever $i < j$. An $n \times n$ matrix $A = (a_{ij})_{n\times n}$ is **upper triangular** if $a_{ij} = 0$ whenever $i > j$.

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If A is upper triangular, lower triangular or diagonal, then $det(A)$ is equal to the product of the diagonal entries.

Using Elementary Row Operations to Find the Determinant

Suppose B is obtained from A by:

1. multiplying a row by a non-zero scalar c, then $det(A) = \frac{1}{c}det(B)$.
2. switching rows, then $det(A) = -det(B)$.
3. adding a multiple of one row to another row, then $det(A) = det(B)$.

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Example: Find the determinant of the following matrix.

$$\begin{pmatrix} 0 & 2 & 0 & -3 \\ 3 & 0 & 2 & 5 \\ -2 & 4 & 0 & 6 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Properties of Determinants

Theorem

If elementary row or column operations lead to one of the following conditions, then the determinant is zero.

1. *an entire row (or column) consists of zeros.*
2. *one row (or column) is a multiple of another row (or column).*

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Let A and B be $n \times n$ matrices and c be a scalar.

1. $det(AB) = det(A)det(B)$
2. $det(cA) = c^n det(A)$
3. $det(A^T) = det(A)$