Math 240: Flux and Stoke's Theorem

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Outline

1. Flux
2. Stokes' Theorem

Integrals of vector fields

$$\int\int_S G(x,y,z)dS =$$

$$\int\int_R G(x,y,f(x,y))\sqrt{1+(f_x(x,y))^2+(f_y(x,y))^2}dA$$

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If F is a vector field and $\mathbf{n}$ is the normal vector to a surface S then the total volume per unit time is the **Flux** and is given by

$$\int\int_S (F \circ \mathbf{n})dS$$

Example Let $T(x, y, z) = x^2 + y^2 + z^2$ model a temperature distribution in 3-space. From physics, heat flow is modeled by $F = -\nabla T$. Find the heat flow out of a sphere of radius *a* centered at the origin.

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$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = f'(t)dt\mathbf{i} + g'(t)dt\mathbf{j} + h'(t)dt\mathbf{k}$$

Stokes' Theorem

Theorem

Let S be an **nice** *oriented surface bounded by a* **nice** *curve C. Let $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a* **nice** *vector field. If C is traversed in the positive direction and $\mathbf{T}$ is the unit tangent vector to C then*

$$\oint_C F \circ d\mathbf{r} = \oint_C (F \circ \mathbf{T}) ds = \int\int_S (curl(F) \circ \mathbf{n}) dS$$

where $\mathbf{n}$ is the unit normal to S in the direction of the orientation of S.

Example: Use Stokes' theorem to evaluate $\oint_C F \circ d\mathbf{r}$ where C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ oriented counterclockwise when viewed from above and $F = (2z + x)\mathbf{i} + (y - z)\mathbf{j} + (x + y)\mathbf{k}$.

Stokes' Theorem

Theorem

Let S be a piecewise smooth oriented surface bounded by a piecewise smooth simple closed curve C. Let $F(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field for which P, Q and R are continuous and have continuous partial derivatives in the region of 3-space containing S. If C is traversed in the positive direction and $\mathbf{T}$ is the unit tangent vector to C then

$$\oint_C F \circ d\mathbf{r} = \oint_C (F \circ \mathbf{T}) ds = \int\int_S (curl(F) \circ \mathbf{n}) dS$$

where $\mathbf{n}$ is the unit normal to S in the direction of the orientation of S.