Math 240: Surface Integrals and Flux

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Outline

Surface Area

Definition

Let $f(x, y)$ be a function with continuous partial derivatives f_x and f_y defined on a region R. The **Area of the surface** $z = f(x, y)$ **over** R is given by

$$\int\int_R \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2} dA.$$

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Example: Calculate the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

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Let G be a scalar function and S be a surface given by the graph of $z = f(x, y)$ over the region R. The **surface integral of G over S** is given by:

$$\int\int_S G(x, y, z)dS =$$

$$\int\int_R G(x, y, f(x, y))\sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}dA$$

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Exercise Find the unit normal vectors to a sphere of radius a.

Integrals of vector fields

Suppose $F(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ models the velocity of a fluid in 3-space.

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Then the volume flowing through a small patch of surface S per unit time is

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Where $\mathbf{n}$ is the normal vector to S and ΔS is the area of the patch of surface.

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The total volume per unit time is the **Flux** and is given by

$$\int\int_S (F \circ \mathbf{n}) dS$$

Example Let $T(x, y, z) = x^2 + y^2 + z^2$ model a temperature distribution in 3-space. From physics, heat flow is modeled by $F = -\nabla T$. Find the heat flow out of a sphere of radius a centered at the origin.