Math 240: More of Green's Theorem and Surface Integrals

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Outline

Today's Goals

1. Investigate the implications of Green's Theorem
2. Be able to calculate surface area.
3. Be able to calculate surface integrals.

Green's Theorem

Theorem (Green's Theorem)

Suppose C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then

$$\oint_C Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA,$$

where C is oriented counterclockwise.

Green's Theorem: Even when it fails it wins.

Example: Evaluate the following integral where C is the positively oriented ellipse $x^2 + 4y^2 = 4$.

$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

The Symmetry Trick

Example: Evaluate the following integral where C is the positively oriented square with vertices $(1,1),(1,-1),(-1,1),(-1,-1)$.

$$\oint_C ye^{x^2}dx + xe^{y^2}dy$$

Methods of Evaluating a Line Integral

1. Parametrization and substitution.
2. Find a Primitive.
3. Use Green's Theorem directly.
4. Use Green's Theorem indirectly to simplify the curve you are integrating along.

Surface Area

Definition

Let $f(x, y)$ be a function with continuous partial derivatives f_x and f_y defined on a region R. The **Area of the surface** $z = f(x, y)$ **over** R is given by

$$\int\int_R \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2} dA.$$

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Example: Calculate the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

Surface Integral

Just like line integral generalizes arc length integral, surface integral generalizes surface area integral.

Definition

Let G be a scalar function and S be a surface given by the graph of $z = f(x, y)$ over the region R. The **surface integral of G over S** is given by:

$$\int\int_S G(x, y, z) ds =$$

$$\int\int_R G(x, y, f(x, y)) \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2} dA$$