Math 240: Double Integrals in Polar Coordinates and Green's Theorem

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Outline

Today's Goals

1. Review the calculation of double integrals in polar coordinates.
2. Review Green's Theorem.

Review Problem

Example For the region R bounded by $y = x$, $x + y = 4$ and $x = 0$ evaluate

$$\int\int_R x + 1 dA$$

Evaluation of Double Integrals in Polar Coordinates

Theorem

Let f be continuous on a region R.
If R is Type PI, then

$$\int\int_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)rdrd\theta$$

If R is Type PII, then

$$\int\int_R f(r,\theta)dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta)rd\theta dr$$

Change of Coordinates

If a region in the plane can be describe in polar coordinates as

$$0 \leq g_1(\theta) \leq r \leq g_2(\theta), \;\; \alpha \leq \theta \leq \beta$$

then we have the following conversion formula

$$\int \int_R f(x, y) dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(rcos(\theta), rsin(\theta)) r dr d\theta$$

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Example Evaluate

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}dydx$$

Green's Theorem

Theorem (Green's Theorem)

Suppose C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then

$$\oint_C Pdx + Qdy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA,$$

where C is oriented counterclockwise.