Math 240: Independence of Path and Double Integrals

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Outline

Today's Goals

1. Understand and apply the tests for path independence.
2. Be able to evaluate of double integrals.

Exact differentials and The Fundamental Theorem of Line Integrals

Definition

The **differential** of a function of two variables $\phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$P(x, y)dx + Q(x, y)dy$ is an **exact differential** if there exists a function $\phi(x, y)$ such that

$$d\phi = P(x, y)dx + Q(x, y)dy$$

The Fundamental Theorem of Line Integrals

Theorem

(Fundamental theorem of Line integrals) Suppose there exists a function $\phi(x, y)$ such that $d\phi = P(x, y)dx + Q(x, y)dy$ and A and B are the endpoints of the path C. Then

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

Test for path independence in 2D

Theorem

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C Pdx + Qdy$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x, y) in the region.

Test for path independence in 3D

Theorem

Let P, Q and R have continuous first partial derivatives in an open simply connected region of space. Then $\int_C Pdx + Qdy + Rdz$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \ \text{and} \ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

for all (x, y, z) in the region.

Intuition of double integrals in the plane

Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

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Additionally, the sides of the object are vertical and the top of the object is the graph of the function $G(x, y)$.

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Additionally, the sides of the object are vertical and the top of the object is the graph of the function $G(x, y)$.

Then the volume of the object is given by

$$\int\int_R G(x, y)dA$$

Where we are integrating with respect to the area of R.

Regions

Definition

A **Type I** region is given by the following formula

$$a \leq x \leq b, \;\; g_1(x) \leq y \leq g_2(x)$$

Definition

A **Type II** region is given by the following formula

$$c \leq y \leq d, \;\; h_1(x) \leq y \leq h_2(x)$$

Evaluation of Double Integrals

Theorem

Let f be continuous on a region R.
If R is Type I, then

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is Type II, then

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example For the region R given by $0 \leq x \leq 2$, $x^2 \leq y \leq 4$ evaluate

$$\int\int_R xe^{y^2} dA$$