Math 240: Line Integrals

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Outline

Review of Last Time

1. Reviewed vector valued functions.
2. Reviewed del, grad, curl and div.

Review Question

Question Let $f(x, y, z) = zx - xy^2$. At the point $(1, 1, 1)$, find the angle between the vector pointing in the direction of fastest increase of $f(x, y, z)$ and the x-axis.

Today's Goals

1. Be able to evaluate line integrals.
2. Understand and apply the tests for path independence.

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Then the total area of the fence is given by

$$\int_C G(x, y)ds$$

where we are integrating with respect to the arc length of C.

Line Integrals in 2D

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Line Integrals in 3D

If G(x,y,z) is a scalar valued function and C is a smooth curve in 3-space defined by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $a \leq t \leq b$ then we can define the following line integrals

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4. $\int_C G(x,y,z)dz = \int_a^b G(f(t),g(t),h(t))h'(t)dt$

Exact differentials and The Fundamental Theorem of Line Integrals

Definition

The **differential** of a function of two variables $\phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$P(x, y)dx + Q(x, y)dy$ is an **exact differential** if there exists a function $\phi(x, y)$ such that

$$d\phi = P(x, y)dx + Q(x, y)dy$$

The Fundamental Theorem of Line Integrals

Theorem

(Fundamental theorem of Line integrals) Suppose there exists a function $\phi(x, y)$ *such that* $d\phi = P(x, y)dx + Q(x, y)dy$ *and* A *and* B *are the endpoints of the path* C. *Then*

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

Test for path independence in 2D

Theorem

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C Pdx + Qdy$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x, y) in the region.

Test for path independence in 3D

Theorem

Let P, Q and R have continuous first partial derivatives in an open simply connected region of space. Then $\int_C P dx + Q dy + R dz$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

for all (x, y, z) in the region.