Math 240: Div, Curl and Line Integrals

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Outline

Review for Last Time

1. Reviewed the definition of vector valued functions and their derivatives.
2. Reviewed the definition of and the calculation of partial derivatives.

Partial Derivative Example

Find $\frac{\partial w}{\partial x}$ if $w = y^{ln(x)} cos(xz)$.

Today's Goals

1. Define and calculate del, grad, curl and div.
2. Review line integrals.

Measuring Vector Fields

Definition

A 3-dimensional vector field is a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ denoted by

$$F(x,y,z) = < f(x,y,z), g(x,y,z), h(x,y,z) >$$

where $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ and scalar valued functions.

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Similarly, a 2-dimensional vector field is of the form $F(x,y) =< f(x,y), g(x,y) >$.

Del and Grad

Motivation: We want an alternative notion of derivative of a function from $\mathbb{R}^2$ or $\mathbb{R}^3$ into $\mathbb{R}$.

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Given a scalar function $f(x, y, z)$ we can form the **gradient of f** using del.

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

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Example: Guess the gradient of $f(x, y, z) = xyz$ at $(1, 1, 1)$ by interpreting the function as volume of a box.

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Divergence measures the tendency of a vector field to expand or contract.

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$$curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$$

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The magnitude of Curl is the intensity of the rotation about a point. The direction of Curl is the axis of maximal rotation about a point.

Curl in Dimension 2

Question: What is curl in dimension 2?

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Theorem (Green's Theorem)

Suppose C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then

$$\oint_C Pdx + Qdy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

where C is oriented counterclockwise.

Domain and Range

It is important to note

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2. div(vector field) = scalar function
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Homework: If F is a 3-dimensional vector field show

$$div(curl(F)) = 0$$