Math 240: Div, Curl and Line Integrals

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University of Pennsylvania

Friday January 13, 2012

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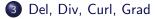
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Review for Last Time

- Reviewed the definition of vector valued functions and their derivatives.
- Reviewed the definition of and the calculation of partial derivatives.

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Image: A matrix

Review

Partial Derivative Example

Find
$$\frac{\partial w}{\partial x}$$
 if $w = y^{\ln(x)} \cos(xz)$.

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- Define and calculate del, grad, curl and div.
- Review line integrals.

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Measuring Vector Fields

Definition

A 3-dimensional vector field is a map from \mathbb{R}^3 to \mathbb{R}^3 denoted by

$$F(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$$

where f(x, y, z), g(x, y, z) and h(x, y, z) and scalar valued functions.

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where f(x, y, z), g(x, y, z) and h(x, y, z) and scalar valued functions.

Similarly, a 2-dimensional vector field is of the form $F(x, y) = \langle f(x, y), g(x, y) \rangle$.

Motivation: We want an alternative notion of derivative of a function from \mathbb{R}^2 or \mathbb{R}^3 into \mathbb{R} .

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Given a scalar function f(x, y, z) we can form the **gradient of f** using del.

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

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 ∇f points in the direction of greatest change of f. **Example:** Guess the gradient of f(x, y, z) = xyz at (1, 1, 1) by interpreting the function as volume of a box.



Motivation: Given a vector field we want to make quantitative the notion of expansion and contraction.

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Definition

The **divergence** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is given by the scalar function

$$div(F) = \nabla \cdot F = \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial Q}{\partial y}\mathbf{j} + \frac{\partial R}{\partial z}\mathbf{k}$$

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Divergence measures the tendency of a vector field to expand or contract.

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Motivation: Given a vector field we want to make quantitative the notion of rotation.

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Curl

Motivation: Given a vector field we want to make quantitative the notion of rotation.

Definition

The curl of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field

$$curl(F) = \nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

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The magnitude of Curl is the intensity of the rotation about a point. The direction of Curl is the axis of maximal rotation about a point.

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Curl in Dimension 2

Question: What is curl in dimension 2?

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Theorem (Green's Theorem)

Suppose C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then $\oint_C Pdx + Qdy = \int \int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA,$

where C is oriented counterclockwise.

Domain and Range

It is important to note

- grad(scalar function) = vector field
- ø div(vector field) = scalar function
- surl(vector field) = vector field

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Homework: If F is a 3-dimensional vector field show

div(curl(F)) = 0