1. (10 pts) Calculate the flux of F across S if $F(x, y, z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$ oriented outward.

A) 0
B) $-\frac{\pi}{4}$
C) $\frac{11\pi}{8}$
D) 3π
E) $\frac{9\pi}{4}$

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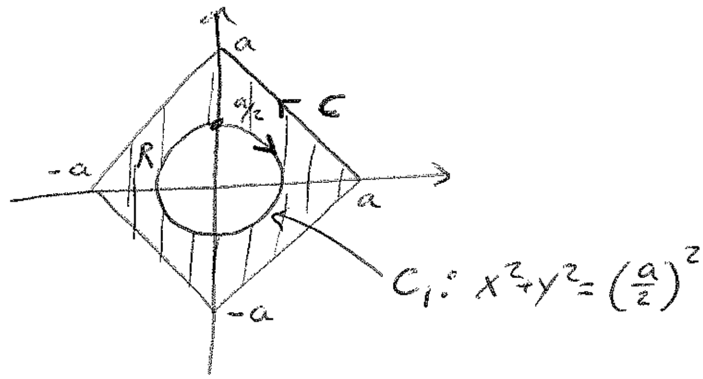
2. (10 pts) Given $F(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{(x^2+y^2)}$ and C is the positively oriented square with vertices $(-a,0),(a,0),(0,-a),(0,a)$, find the following

$$lim_{a\to 0} \oint_C F \circ dr$$



A) 0
B) 2π
B) $\frac{1}{2}$
C) $-\pi$
D) ∞
E) DNE

By Green's theorem

$$\int_{C \cup C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \iint_R \frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x} - \frac{\partial\left(\frac{-y}{x^2+y^2}\right)}{\partial y}\,dx\,dy$$

$$= \iint_R 0\,dx\,dy = 0$$

So, $$\lim_{a\to 0} \oint_C F\circ dr = \lim_{a\to 0}\int_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \overset{\lim_{a\to 0}}{=} -\int_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

Parametrize $-C_1 : \langle \frac{a}{2}\cos(t), \frac{a}{2}\sin(t)\rangle \quad 0 \le t \le 2\pi$

$$\lim_{a\to 0} \oint F\circ dr = \lim_{a\to 0} \int_0^{2\pi} \frac{-\frac{a}{2}\sin(t)}{\left(\frac{a}{2}\right)^2}\left(-\frac{a}{2}\sin(t)\right) + \frac{\frac{a}{2}\cos(t)}{\left(\frac{a}{2}\right)^2}\frac{a}{2}\cos(t)\,dt$$

$$= \lim_{a\to 0} \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt$$

$$= \lim_{a\to 0} 2\pi$$

$$= 2\pi$$

3. (10 pts) Let $f(x)$ be the solution to the following IVP.

$$y'' - 4y' + 4y = e^{2x} \quad y(0) = 1 \quad y'(0) = 1$$

What is $f(1)$.

A) 1
B) $\frac{e^2}{2}$
B) $\frac{1}{2}$
C) e^2
D) $\frac{-e^2}{2}$
E) $\frac{-e^2}{4}$

Step 1 Solve homogeneous eq.

$$y'' - 4y' + 4y = 0$$

Aux. Eq. $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2 \quad m=2$$

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2: Use Undetermined Coef. to guess y_p.

Guess $y_p = Ae^{2x}$ Bad! (repeated in homogeneous)

$y_p = Axe^{2x}$ Bad! (repeated in homogeneous)

$y_p = Ax^2e^{2x}$ Good!

$$y_p' = 2Ax^2e^{2x} + 2Axe^{2x}$$

$$y_p'' = 4Ax^2e^{2x} + 4Axe^{2x} + 4Axe^{2x} + 2Ae^{2x}$$

Substitute in:

$$\cancel{4Ax^2e^{2x}} + \cancel{4Axe^{2x}} + \cancel{4Axe^{2x}} + 2Ae^{2x} - \cancel{8Ax^2e^{2x}} - \cancel{8Axe^{2x}} + \cancel{4Ax^2e^{2x}} = e^{2x}$$

$$2Ae^{2x} = e^{2x}$$

$$\boxed{A = \tfrac{1}{2}} \quad y_p = \tfrac{1}{2}x^2e^{2x}$$

$$y_g = c_1 e^{2x} + c_2 x e^{2x} + \tfrac{1}{2}x^2e^{2x} \quad y(0) = 1,\ y'(0) = 1$$

$$c_1 = 1 \qquad 2c_1 + c_2 = 1 \Rightarrow c_2 = -1$$

$$y_g(t) = e^{2x} + (-1)xe^{2x} + \tfrac{1}{2}x^2e^{2x}$$

4. (10 pts) Please circle "T" for true or "F" for false in the space provided to the left of the following statements. You **DO NOT** need to justify your answer for full credit.

(T F) Every 2×2 diagonalizable matrix with repeated eigenvalue is a diagonal matrix.

(T F) There is a vector field F such that $curl(F) =< x, y, z >$.

(T F) If $det(A) = 0$, then the system $Ax = 0$ has ∞-many solutions.

(T F) If y_1 and y_2 are solutions to a non-homogeneous linear differential equation, then $y_1 + y_2$ is also a solution.

(T F) If A and B are square matrixes such that $AB^2 = I$, then B is invertible.

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5. (10 pts) What are the possible eigenvalues of an $n \times n$ matrix A if A satieties the following matrix equation?

$$A - 2I = -A^2$$

$A)$ $0, 1$
$B)$ $0, 2$
$C)$ $0, 1, -2$
$D)$ $1, -2$
$E)$ $1, -3$

does not have an eigen value of zero.

Is A invertible? (Yes No)

Recall, If λ is an eigen value, there is a vector v s.t. $Av = \lambda v$.

Let v be an eigen vector

$$(A - 2I)v = -A^2 v$$

$$Av - 2v = -A^2 v$$

$$\lambda v - 2v = -A(\lambda v)$$

$$\lambda v - 2v = -\lambda^2 v$$

$$\lambda^2 v + \lambda v - 2v = \vec{0}$$

$$(\lambda^2 + \lambda - 2)v = \vec{0}$$

Since $\vec{v} \neq \vec{0}$ then $\lambda^2 + \lambda - 2 = 0$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\lambda = 1 \text{ or } -2$$

6. (10pts) Find the general solution to the system of linear ODE

$$\frac{d}{dt}u(x) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} u(x), \quad where \quad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}$$

$u(x) =$ $c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^x + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2x} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x e^{2x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2x} \right)$

Step 1: Find E. val.

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 2 \text{ or } 1$$

Step 2: Find E. Vec.

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x = 0$, $y = 0$ $z = $ free

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x = z$ $y = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step 3: Find generalized E. vec.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$P_2 = 1$ & $P_1 - P_3 = 1$

So, a solution is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Hence, the general sol. is $u(x) = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^x + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2x} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cancel{x} e^{2x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2x} \right)$

7.(10pts)
(a) Give an example of a 3×3 matrix A which has only two eigenvalues and A is *not* diagonalizable. In other words, there does not exist an invertible 3×3 matrix C such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix. **Justify your answer.**

$A =$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

E. val.
$\lambda = 2, 3$

E. vec. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(b) Give an example of a 3×3 matrix B which has only two eigenvalues and B is diagonalizable. **Justify your answer.**

$A =$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

e. val
$\lambda = 2, 3$

e. vec.
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

8.(10pts) Find A^{201} if

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$$

$A^{201} = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$.

Trick 1: A is diagnolizable

Step 1: Find e.val $\begin{vmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$(3-\lambda)(-3-\lambda) + 8 = 0$

$-9 + \lambda^2 + 8 = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

Step 2: Find e. ~~val~~ vec.

$\lambda = 1$

$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = y$ $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

$\begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x = y$ $\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $P^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$A^{201} = \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\right)^{201} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{201}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= A$

Trick 2: Notice that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9.(10pts)Produce a basis of all skew symmetric 3×3 matrices that includes the following matrix. Show that set you produce is a basis.

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Claim: This set spans

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & (a-c)+c & (b-c)+c \\ (-a+c)-c & 0 & c \\ -b+c-c & -c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a-c & 0 \\ -a+c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b-c \\ 0 & 0 & 0 \\ -b+c & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c & c \\ -c & 0 & c \\ -c & -c & 0 \end{bmatrix}$$

$$= (a-c)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (b-c)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Claim: This set is L.I

Examine $c_1\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & c_1+c_3 & c_1+c_2 \\ -c_1-c_3 & 0 & c_1 \\ -c_1-c_2 & -c_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $c_1 = 0 \quad c_1 + c_2 = 0 \quad c_1 + c_3 = 0$

Hence $c_1 = 0, \ c_2 = 0, \ c_3 = 0.$

10.(10pts) Find matrices A, B, C, D with the following properties

1. A is invertible and diagonalizable
2. B is invertible and NOT diagonalizable
3. C is NOT invertible and diagonalizable
4. D is NOT invertible and NOT diagonalizable

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$