Math 240: Line Integrals

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Friday Sept. 7, 2012

Outline

1. Review
2. Today's Goals
3. Line Integrals

Curl

Motivation: Given a vector field we want to make quantitative the notion of rotation.

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Definition

The **curl** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the **vector field**

$$curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$$

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The magnitude of Curl is the intensity of the rotation about a point. The direction of Curl is the axis of maximal rotation about a point.

Domain and Range

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Homework: If F is a 3-dimensional vector field show

$$div(curl(F)) = 0$$

Today's Goals

1. Be able to evaluate line integrals.
2. Be able to apply Green's theorem.

Intuition of line integrals in the plane

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Suppose we want to build a fence along a path $\mathbf{p}$ in the plane.

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Then the total area of the fence is given by

$$\int_{\mathbf{p}} fds.$$

Definition

The **scalar line integral** of f along the differentiable path $\mathbf{p} : [a, b] \to \mathbb{R}^n$ is

$$\int_a^b f(\mathbf{p}(t))||\mathbf{p}'(t)||dt$$

We denote this integral by $\int_{\mathbf{p}} fds$.

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Other scalar line integral formulas for $\mathbf{p} =< x(t), y(t), ... >$:

$$\int_{\mathbf{p}} fdx = \int_a^b f(\mathbf{p}(t))x'(t)dt$$

$$\int_{\mathbf{p}} fdy = \int_a^b f(\mathbf{p}(t))y'(t)dt$$

Definition

The **vector line integral** of a vector field $\mathbf{F}$ along the differentiable path $\mathbf{p} : [a, b] \to \mathbb{R}^n$ is

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{p}(t)) \cdot \mathbf{p}'(t) dt.$$

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If **F** is a force field, then this integral measures the work done by **F** on a particle moving along **p**.

Green's Theorem

Theorem (Green's Theorem)

Let D be a closed, bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$. Orient the curves of C so that D is on the left as one traverses C. Let $F(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a vector field such that M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on D, then

$$\oint_C Mdx + Ndy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$$