Math 240: Stokes's Theorem and Gauss's Theorem

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Outline

Today's Goals

1. Be able to apply Stokes's Theorem
2. Be able to apply Gauss's Theorem

Stokes' Theorem

Theorem

Let S be a **nice** *oriented surface in R^3. Suppose that ∂S is oriented consistently with S. Let* **F** *be a vector field of class C^1 defined on S. Then*

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Example: Find the flux of the curl of **F** across the upward oriented surface given by $z = 4 - x^2 - y^2$ and $z \geq 0$ where $\mathbf{F} =< z^2 - y, x + z, z^3 >$.

Stokes' Theorem

Theorem

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Example: Let $F = < y^2, 2z + x, 2y^2 >$. Find a plane $ax + by + cz = 0$ such that $\oint_C F \circ dr = 0$ for every smooth simple closed curve C in the plane.

Gauss's Theorem

Theorem

Let D be a bounded solid region in R^3 whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes D. Then

$$\int\int_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{D} \nabla \cdot \mathbf{F} dV.$$

Gauss's Theorem

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Example Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$ and $z = 1$.

Gauss's Theorem

Theorem

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Example: Use the divergence theorem to evaluate $\int\int_S (\mathbf{F} \cdot \mathbf{n}) dS$ where $\mathbf{F} = < x + y, z, z - x >$ and S is the boundary of the region between $z = 9 - x^2 - y^2$ and the xy-plane.

Stokes's Theorem

Theorem

Let S be a bounded, piecewise smooth, oriented surface in R^3. Suppose that ∂S consists of finitely many piecewise C^1, simple, closed curves each of which is oriented consistently with S. Let $\mathbf{F}$ be a vector field of class C^1 defined on S. Then

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

Gauss's Theorem

Theorem

Let D be a bounded solid region in R^3 whose boundary ∂D consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normals that point away from D. Let $\mathbf{F}$ be a vector field of class C^1 whose domain includes D. Then

$$\int\int_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_D \nabla \cdot \mathbf{F} dV.$$