Math 240: Surface Integrals

Ryan Blair

University of Pennsylvania

Monday Sept. 17, 2012

Outline

Today's Goals

1. Review of surface area for a parameterized surfaces.
2. Be able to evaluate Surface Integrals.

Surface Area of a Parameterized Surface

Given a parametrization $\mathbf{X} : D \to \mathbb{R}^3$ such that $X(s, t) = (x(s, t), y(s, t), z(s, t))$.

The surface area of $S = \mathbf{X}(D)$ is equal to

$$\int\int_D \| T_s \times T_t \| dsdt$$

Scalar Surface Integrals

Definition

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where D is a region in $\mathbb{R}^2$. Let $f : \mathbf{X}(D) \to \mathbb{R}$ be a continuous function. Then the **scalar surface integral** of f along $\mathbf{X}$ is

$$\int\int_X f dS = \int\int_D f(\mathbf{X}(s,t)) \parallel T_s \times T_t \parallel dsdt$$

Example Find $\int\int_S x^2 + y^2 dS$, where S is the outward oriented lateral surface of the cylinder of radius a and height h whose axis is the z-axis.

Vector Surface Integrals

Definition

Let $\mathbf{X} : D \to \mathbb{R}^3$ be a smooth parametrized surface, where D is a region in $\mathbb{R}^2$. Let $\mathbf{F}(x, y, z)$ be a continuous $f : \mathbf{X}(D) \to \mathbb{R}$ be a continuous function. Then the **vector surface integral** (or **Flux**) of $\mathbf{F}$ along $\mathbf{X}$ is

$$\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t) dsdt$$

where $\mathbf{N}(s,t) = T_s \times T_t$.

ExampleFind the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface S consisting of the triangular region of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Use an upward-pointing normal to orient S.

Orientation

Definition

An **orientable** surface has two sides that can be painted red and blue resp.

Definition

If a parameterized surface S is orientable, then an **orientation** is a choice of one of two normal vectors.

$$N_1 = T_s \times T_t$$

or

$$N_2 = T_t \times T_s = -N_1$$