Math 240: Parameterized Surfaces

Ryan Blair

University of Pennsylvania

Friday Sept. 14, 2012

Outline

Today's Goals

1. Finish Green's Theorem.
2. Be able to find the surface area of parameterized surfaces.

Green's Theorem: Even when it fails it wins.

Example: Evaluate the following integral where C is the positively oriented ellipse $x^2 + 4y^2 = 4$.

$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

Theorem (Green's Theorem)

Let D be a closed, bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$. Orient the curves of C so that D is on the left as one traverses C. Let $F(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a vector field such that M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on D, then

$$\oint_C Mdx + Ndy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$$

Methods of Evaluating a Scalar Line Integral in the Plane

1. Parametrization and substitution.
2. Use the fundamental theorem of line integrals.
3. Use Green's Theorem directly.
4. Use Green's Theorem indirectly to simplify the curve you are integrating along.

Parameterized Surface

Definition

A **parameterized surface** is a continuous, one-to-one map from a region in the plane into $\mathbb{R}^3$.
More precisely, $\mathbf{X} : D \to \mathbb{R}^3$ such that
$X(s,t) = (x(s,t), y(s,t), z(s,t))$.

Tangent and Normal Vectors

Given a parametrization $\mathbf{X} : D \to \mathbb{R}^3$ such that $X(s,t) = (x(s,t), y(s,t), z(s,t))$.

Definition

The **tangent vector with respect to** s is
$\mathbf{T}_s(s,t) = \frac{\partial \mathbf{X}}{\partial s}(s,t) = \frac{\partial x}{\partial s}(s,t)\mathbf{i} + \frac{\partial y}{\partial s}(s,t)\mathbf{j} + \frac{\partial z}{\partial s}(s,t)\mathbf{k}$.
Similarly, $\mathbf{T}_t(s,t) = \frac{\partial \mathbf{X}}{\partial t}(s,t) = \frac{\partial x}{\partial t}(s,t)\mathbf{i} + \frac{\partial y}{\partial t}(s,t)\mathbf{j} + \frac{\partial z}{\partial t}(s,t)\mathbf{k}$.

Tangent and Normal Vectors

Given a parametrization $\mathbf{X} : D \to \mathbb{R}^3$ such that $X(s,t) = (x(s,t), y(s,t), z(s,t))$.

Definition

The **tangent vector with respect to** s is
$\mathbf{T}_s(s,t) = \frac{\partial \mathbf{X}}{\partial s}(s,t) = \frac{\partial x}{\partial s}(s,t)\mathbf{i} + \frac{\partial y}{\partial s}(s,t)\mathbf{j} + \frac{\partial z}{\partial s}(s,t)\mathbf{k}$.
Similarly, $\mathbf{T}_t(s,t) = \frac{\partial \mathbf{X}}{\partial t}(s,t) = \frac{\partial x}{\partial t}(s,t)\mathbf{i} + \frac{\partial y}{\partial t}(s,t)\mathbf{j} + \frac{\partial z}{\partial t}(s,t)\mathbf{k}$.

Definition

The **Normal vector** is $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t)$

Surface Area of a Parameterized Surface

Given a parametrization $\mathbf{X} : D \rightarrow \mathbb{R}^3$ such that $X(s,t) = (x(s,t), y(s,t), z(s,t))$.

The surface area of $S = \mathbf{X}(D)$ is equal to

$$\int\int_D \parallel T_s \times T_t \parallel dsdt$$