Math 240: Double integrals and Green's Theorem

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Wednesday Sept. 12, 2012

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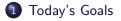
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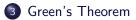
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2 Double Integrals



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- Be able to evaluate double integrals.
- Be able to apply Green's Theorem. 2

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Intuition of double integrals in the plane

Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

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Additionally, the sides of the object are vertical and the top of the object is the graph of the function G(x, y).

Then the volume of the object is given by

$$\int \int_R G(x,y) dA.$$

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Definition

A Type I region is given by the following formula

$$a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)$$

Definition

A Type II region is given by the following formula

$$c \leq y \leq d$$
, $h_1(x) \leq y \leq h_2(x)$

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Evaluation of Double Integrals

Theorem

Let f be continuous on a region R. If R is Type I, then

$$\int \int_{R} f(x,y) dA = \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x,y) dy dx$$

If R is Type II, then

$$\int \int_{R} f(x,y) dA = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x,y) dx dy$$

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Example For the region R given by $0 \le x \le 2$, $x^2 \le y \le 4$ evaluate

$$\int \int_R x e^{y^2} dA$$

Green's Theorem

Theorem (Green's Theorem)

Let D be a closed, bounded region in \mathbb{R}^2 with boundary $C = \partial D$. Orient the curves of C so that D is on the left as one traverses C. Let $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field such that M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on D, then

$$\oint_C M dx + N dy = \int \int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dx dy.$$

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Definition

Given a region R in \mathbb{R}^n , the **boundary** of R, denoted ∂R , is the collection of all points that are adjacent to both R and the complement of R.

Making Impossible Line Integrals Possible

Example: Evaluate the following line integral on the triangle with vertices (-1, 1), (0, 1) and (0, 0).

$$\oint_C e^{x^2} dx + 2tan^{-1}(x) dy$$

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Theorem (Green's Theorem)

If everything is nice, then

$$\oint_C M dx + N dy = \int \int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dx dy.$$

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Green's theorem holds for regions with multiple boundary curves

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Green's theorem holds for regions with multiple boundary curves **Example:**Let C be the positively oriented boundary of the annular region between the circle of radius 1 and the circle of radius 2. Evaluate

$$\oint_C (4x^2 - y^3) dx + (x^3 + y^3) dy$$

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Green's Theorem: Even when it fails it wins.

Example: Evaluate the following integral where *C* is the positively oriented ellipse $x^2 + 4y^2 = 4$.

$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

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Methods of Evaluating a Scalar Line Integral in the Plane

- Parametrization and substitution.
- Find a Primitive.
- Use Green's Theorem directly.
- Use Green's Theorem indirectly to simplify the curve you are integrating along.