Math 240: Double integrals and Green's Theorem

Ryan Blair

University of Pennsylvania

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Outline

Today's Goals

1. Be able to evaluate double integrals.
2. Be able to apply Green's Theorem.

Intuition of double integrals in the plane

Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

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Intuition of double integrals in the plane

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Additionally, the sides of the object are vertical and the top of the object is the graph of the function $G(x, y)$.

Then the volume of the object is given by

$$\int\int_R G(x, y)dA.$$

Regions

Definition

A **Type I** region is given by the following formula

$$a \leq x \leq b, \;\; g_1(x) \leq y \leq g_2(x)$$

Definition

A **Type II** region is given by the following formula

$$c \leq y \leq d, \;\; h_1(x) \leq y \leq h_2(x)$$

Evaluation of Double Integrals

Theorem

Let f be continuous on a region R.
If R is Type I, then

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is Type II, then

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example For the region R given by $0 \leq x \leq 2$, $x^2 \leq y \leq 4$ evaluate

$$\int\int_R xe^{y^2}\, dA$$

Green's Theorem

Theorem (Green's Theorem)

Let D be a closed, bounded region in $\mathbb{R}^2$ with boundary $C = \partial D$. Orient the curves of C so that D is on the left as one traverses C. Let $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be a vector field such that M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on D, then

$$\oint_C Mdx + Ndy = \int\int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy.$$

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Definition

Given a region R in $\mathbb{R}^n$, the **boundary** of R, denoted ∂R, is the collection of all points that are adjacent to both R and the complement of R.

Making Impossible Line Integrals Possible

Example: Evaluate the following line integral on the triangle with vertices $(-1,1)$, $(0,1)$ and $(0,0)$.

$$\oint_C e^{x^2}dx + 2tan^{-1}(x)dy$$

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Green's Theorem: Even when it fails it wins.

Example: Evaluate the following integral where C is the positively oriented ellipse $x^2 + 4y^2 = 4$.

$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

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Methods of Evaluating a Scalar Line Integral in the Plane

1. Parametrization and substitution.
2. Find a Primitive.
3. Use Green's Theorem directly.
4. Use Green's Theorem indirectly to simplify the curve you are integrating along.