# Math 240: Line Integrals and Path Independence

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Monday Sept. 10, 2012

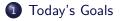
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Math 240: Line Integrals

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## 2 Line Integrals



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- Be able to evaluate line integrals.
- Be able to evaluate double integrals.

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Last time we saw scalar line integrals.

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## Definition

The **vector line integral** of a vector field **F** along the differentiable path  $\mathbf{p} : [a, b] \to \mathbb{R}^n$  is

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{a}^{b} \mathbf{F}(\mathbf{p}(t)) \cdot \mathbf{p}'(t) dt.$$

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If  $\mathbf{F}$  is a force field, then this integral measures the work done by  $\mathbf{F}$  on a particle moving along  $\mathbf{p}$ .

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### Definition

A continuous vector field **F** has path-independent line integrals if

$$\int_{p_1} \mathbf{F} \cdot d\mathbf{s} = \int_{p_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two paths having the same initial and terminal points.

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#### Theorem

Let  ${\bf F}$  be a continuous vector field. Then  ${\bf F}$  has path-independent line integrals if

$$\int_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for all simple closed curves C.

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# Fundamental Theorem of line integrals

#### Theorem

Let **F** be a **nice** vector field. Then  $F = \nabla f$  for a **nice** function f if and only if for any **nice** path C with initial point A and terminal point B, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A).$$

## When is $\mathbf{F} = \nabla f$ ?

#### Theorem

Let **F** be a vector field of class C1 whose domain is a simplyconnected region R in either  $R^2$  or  $R^3$ . Then  $F = \nabla f$  if and only if  $\nabla \times \mathbf{F} = curl(\mathbf{F}) = 0$  at all points of R.

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Simply connected means that every loop can be continuously deformed to a point. i.e. No holes.

## Intuition of double integrals in the plane

Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

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# Intuition of double integrals in the plane

Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

Additionally, the sides of the object are vertical and the top of the object is the graph of the function G(x, y).

Then the volume of the object is given by

$$\int \int_R G(x,y) dA.$$



## Definition

A Type I region is given by the following formula

$$a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)$$

### Definition

A Type II region is given by the following formula

$$c \leq y \leq d$$
,  $h_1(x) \leq y \leq h_2(x)$ 

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# **Evaluation of Double Integrals**

## Theorem

Let f be continuous on a region R. If R is Type I, then

$$\int \int_{R} f(x,y) dA = \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x,y) dy dx$$

If R is Type II, then

$$\int \int_{R} f(x,y) dA = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x,y) dx dy$$

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## **Example** For the region R given by $0 \le x \le 2$ , $x^2 \le y \le 4$ evaluate

$$\int \int_R x e^{y^2} dA$$