Math 240: Line Integrals and Path Independence

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Outline

Today's Goals

1. Be able to evaluate line integrals.
2. Be able to evaluate double integrals.

Last time we saw scalar line integrals.

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Definition

The **vector line integral** of a vector field $\mathbf{F}$ along the differentiable path $\mathbf{p} : [a, b] \to \mathbb{R}^n$ is

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{p}(t)) \cdot \mathbf{p}'(t) dt.$$

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If $\mathbf{F}$ is a force field, then this integral measures the work done by $\mathbf{F}$ on a particle moving along $\mathbf{p}$.

Definition

A continuous vector field **F** has path-independent line integrals if

$$\int_{p_1} \mathbf{F} \cdot d\mathbf{s} = \int_{p_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two paths having the same initial and terminal points.

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Theorem

Let **F** *be a continuous vector field. Then* **F** *has path-independent line integrals if*

$$\int_{C} \mathbf{F} \cdot d\mathbf{s} = 0$$

for all simple closed curves C.

Fundamental Theorem of line integrals

Theorem

Let **F** *be a* **nice** *vector field. Then* $F = \nabla f$ *for a* **nice** *function* f *if and only if for any* **nice** *path* C *with initial point* A *and terminal point* B*, then*

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A).$$

When is $\mathbf{F} = \nabla f$?

Theorem

Let $\mathbf{F}$ *be a vector field of class* $C1$ *whose domain is a* **simplyconnected** *region* R *in either* R^2 *or* R^3*. Then* $F = \nabla f$ *if and only if* $\nabla \times \mathbf{F} = curl(\mathbf{F}) = 0$ *at all points of* R*.*

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Simply connected means that every loop can be continuously deformed to a point. i.e. No holes.

Intuition of double integrals in the plane

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Suppose we want to find the volume of an object with a flat base in the shape of the region R in the plane.

Additionally, the sides of the object are vertical and the top of the object is the graph of the function $G(x, y)$.

Then the volume of the object is given by

$$\int\int_R G(x, y)dA.$$

Regions

Definition

A **Type I** region is given by the following formula

$$a \leq x \leq b, \;\; g_1(x) \leq y \leq g_2(x)$$

Definition

A **Type II** region is given by the following formula

$$c \leq y \leq d, \;\; h_1(x) \leq y \leq h_2(x)$$

Evaluation of Double Integrals

Theorem

Let f be continuous on a region R.
If R is Type I, then

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is Type II, then

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example For the region R given by $0 \leq x \leq 2$, $x^2 \leq y \leq 4$ evaluate

$$\int \int_R x e^{y^2} \, dA$$