Math 240: Spring-Mass Systems

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Outline

1. Today's Goals
2. Spring/Mass Systems with Damped Motion
3. Spring/Mass Systems with Forced Oscillations

Today's Goals

1. Learn how to solve spring/mass systems.

Spring/Mass Systems with Damped Motion

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$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

is now our model, where m is the mass, k is the positive spring constant, c is the positive damping constant and $y(t)$ is the position of the mass at time t.

Changing Variables

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and the roots of the Aux. Equation become

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega_0^2} \text{ and } m_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2}$$

Case 1: Overdamped

If $\lambda^2 - \omega_0^2 > 0$ the system is **overdamped** since c is large when compared to k. In this case the solution is

$$y = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega_0^2}t} + c_2 e^{\sqrt{\lambda^2 - \omega_0^2}t}).$$

Case 2: Critically Damped

If $\lambda^2 - \omega_0^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$y = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3: Underdamped

If $\lambda^2 - \omega_0^2 < 0$ the system is **underdamped** since k is large when compared to c. In this case the solution is

$$y = e^{-\lambda t}(c_1 cos(\sqrt{\omega_0^2 - \lambda^2} t) + c_2 sin(\sqrt{\omega_0^2 - \lambda^2} t))$$

.

Example

A 4 meter spring measures 8 meters long after a force of 16 newtons acts to it. A mass of 2 kilograms is attached to the spring. The medium through which the mass moves offers a damping force equal to 2 times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 meters/sec.

Forced Oscillations

When an external force $f(t)$ acts on the mass on a spring, the equation for our model of motion becomes

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or in the language of λ and ω,

$$\frac{d^2y}{dt^2} + 2\lambda\frac{dy}{dt} + \omega^2 y = \frac{f(t)}{m}$$

Example

When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest at equilibrium position. Starting at $t = 0$ a force of $f(t) = 65e^{-2t}$ is applied to the system. In the absence of damping, find the equation of motion.

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What is the amplitude of the oscillation after a very long time?

Resonance

When a mass of 5 kg is attached to a spring whose constant is 125 N/m, it comes to rest at equilibrium position. Starting at $t = 0$ a force of $f(t) = 2cos(5t)$ is applied to the system. In the absence of damping, find the equation of motion.

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