Math 240: Spring-Mass Systems

Ryan Blair

University of Pennsylvania

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Outline

Today's Goals

1. Learn how to solve spring/mass systems.

The Method of Undetermined Coefficients

To solve a nonhomogeneous constant coefficient linear differential equation

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1. Step 1: Solve the associated homogeneous equation.
2. Step 2: Find a particular solution by making a guess based on $g(x)$.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.

Spring-Mass Systems with Undamped Motion

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Hooke's Law: The spring exerts a restoring force F_s opposite to the direction of elongation and proportional to the amount of elongation.

$$F_s = -kL_0$$

Newton's Second Law

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Assuming free motion, **Newton's Second Law** states

$$m\frac{d^2y}{dt^2} = -k(L_0 + y) + mg = -ky$$

Solutions to Undamped Spring Equation

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If $\omega_0^2 = \frac{k}{m}$ then the solutions are
$y(t) = c_1 cos(\omega_0 t) + c_2 sin(\omega_0 t)$.
Example: A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.

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$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

is now our model, where m is the mass, k is the positive spring constant, c is the positive damping constant and $y(t)$ is the position of the mass at time t.

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and the roots of the Aux. Equation become

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega_0^2} \textit{ and } m_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2}$$

Case 1: Overdamped

If $\lambda^2 - \omega_0^2 > 0$ the system is **overdamped** since c is large when compared to k. In this case the solution is

$$y = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega_0^2}t} + c_2 e^{\sqrt{\lambda^2-\omega_0^2}t}).$$

Case 2: Critically Damped

If $\lambda^2 - \omega_0^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$y = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3: Underdamped

If $\lambda^2 - \omega_0^2 < 0$ the system is **underdamped** since k is large when compared to c. In this case the solution is

$$y = e^{-\lambda t}(c_1 cos(\sqrt{\omega_0^2 - \lambda^2} t) + c_2 sin(\sqrt{\omega_0^2 - \lambda^2} t))$$

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Example

A 4 meter spring measures 8 meters long after a force of 16 newtons acts to it. A mass of 8 kilograms is attached to the spring. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 meters/sec.