Math 240: Systems of Differential Equations

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Outline

Today's Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Define systems of differential equations
2. Develop the notion of Linear Independence.
3. Develop the notion of General Solution.

Diagonalizability Theorems

Theorem

A $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

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Note: Not all diagonalizable matrices have n distinct eigenvalues.

Linear systems

Definition

A **system of differential equations** is of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

Where $A(t)$ is an $n \times n$ matrix of functions, both $\mathbf{x}(t)$ and $\mathbf{b}(t)$ are $n \times 1$ matrices of functions and $\mathbf{x}'(t)$ is the $n \times 1$ matrix of derivatives of entries in $\mathbf{x}(t)$.
A **solution**

Example:

$$x_1'(t) = cos(t)x_1(t) - sin(t)x_2(t)$$
$$x_2'(t) = sin(t)x_1(t) + cos(t)x_2(t)$$

Solutions

Definition

Given a system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ a **solution vector** is an $n \times 1$ column matrix with differentialable functions as entries that satisfies the system.

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The following is an **initial value problem** for a first order system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ and $\mathbf{x}(t_0) = \mathbf{x}_0$

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Note: As long as everything in sight is continuous on an interval I containing t_0, then there exists a unique solution to the above IVP.

Solutions to Linear Systems

Let $V_n(t)$ be the set of $n \times 1$ matrices with entries consisting of functions. $V_n(t)$ is a vector space under the natural operations of vector addition and scalar multiplication.

Theorem

Solutions to **homogeneous** *systems of differential equations of the form*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

form a vector subspace of $V_n(t)$.

Definition

Given an $n \times n$ homogeneous solution $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, a fundamental solution is a set of n linearly independent solutions.

The Wronskian

Theorem

Let X_1, X_2, ..., X_n be n solution vectors to a homogeneous system on an interval I. They are linearly independent if and only if their **Wronskian** *is non-zero for every t in the interval.*

Guessing a Solution

Given a constant coefficient, linear, homogeneous, first-order system

$$\mathbf{x'=Ax}$$

our intuition prompts us to guess a solution vector of the form

$$\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t}$$

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Hence, we can find such a solution vector iff K is an eigenvector for A with eigenvalue λ.

General Solution with Distinct Real Eigenvalues

Theorem

Let λ_1, λ_2, *...*, λ_n *be n* **distinct** *real eigenvalues of the* $n \times n$ *coefficient matrix* **A** *of the homogeneous system* **X'=AX**, *and let* $\mathbf{K}_1$, $\mathbf{K}_2$, *...* , $\mathbf{K}_n$ *be the corresponding eigenvectors. Then the general solution on* $(-\infty, \infty)$ *is*

$$\mathbf{X} = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t} + ... + c_n \mathbf{K}_n e^{\lambda_n t}$$

where the c_i *are arbitrary constants.*

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Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$