Math 240: Systems of Differential Equations

Ryan Blair

University of Pennsylvania

Friday November 9, 2012

Ryan Blair (U Penn)

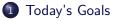
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Outline



- 2 Diagonalizability Theorems
- 3 Linear Systems
- 4 Solutions to Linear Systems
- Distinct Eigenvalues

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Combine linear algebra and differential equations to study systems of differential equations.

- Define systems of differential equations
- **2** Develop the notion of Linear Independence.
- Solution Develop the notion of General Solution.

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Diagonalizability Theorems

Theorem

A $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Theorem

If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable.

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Note:Not all diagonalizable matrices have n distinct eigenvalues.

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Linear systems

Definition

A system of differential equations is of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

Where A(t) is an $n \times n$ matrix of functions, both $\mathbf{x}(t)$ and $\mathbf{b}(t)$ are $n \times 1$ matrices of functions and $\mathbf{x}'(t)$ is the $n \times 1$ matrix of derivatives of entries in $\mathbf{x}(t)$. A solution

Example:

$$\begin{aligned} x_1'(t) &= \cos(t) x_1(t) - \sin(t) x_2(t) \\ x_2'(t) &= \sin(t) x_1(t) + \cos(t) x_2(t) \end{aligned}$$

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Solutions

Definition

Given a system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ a solution vector is an $n \times 1$ column matrix with differentialable functions as entries that satisfies the system.

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The following is an **initial value problem** for a first order system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ and $\mathbf{x}(t_0) = \mathbf{x}_0$

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Note: As long as everything in sight is continuous on an interval I containing t_0 , then there exists a unique solution to the above IVP.

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Solutions to Linear Systems

Let $V_n(t)$ be the set of $n \times 1$ matrices with entries consisting of functions. $V_n(t)$ is a vector space under the natural operations of vector addition and scalar multiplication.

Theorem

Solutions to homogeneous systems of differential equations of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

form a vector subspace of $V_n(t)$.

Definition

Given an $n \times n$ homogeneous solution $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, a fundamental solution is a set of n linearly independent solutions.

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The Wronskian

Theorem

Let X_1 , X_2 , ..., X_n be n solution vectors to a homogeneous system on an interval I. They are linearly independent if and only if their **Wronskian** is non-zero for every t in the interval.

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Guessing a Solution

Given a constant coefficient, linear, homogeneous, first-order system

our intuition prompts us to guess a solution vector of the form

$$\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t}$$

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Hence, we can find such a solution vector iff K is an eigenvector for A with eigenvalue λ .

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General Solution with Distinct Real Eigenvalues

Theorem

Let λ_1 , λ_2 , ..., λ_n be n distinct real eigenvalues of the $n \times n$ coefficient matrix **A** of the homogeneous system **X'=AX**, and let K_1 , K_2 , ..., K_n be the corresponding eigenvectors. Then the general solution on $(-\infty, \infty)$ is

$$\mathbf{X} = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{K}_n e^{\lambda_n t}$$

where the c_i are arbitrary constants.

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Exercise: Solve the linear system X' = AX if

$$A = \left(\begin{array}{cc} -1 & 2\\ -7 & 8 \end{array}\right)$$

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