Math 240: Eigenvalues and Linear Transformations of $\mathbb{R}^2$

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Outline

1. Invertible Linear Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$
2. Eigenvalue and Eigenvector

Today's Goals

1. Know how to decompose linear transformations of $\mathbb{R}^2$ into stretches, reflections and shears.
2. Know how to calculate eigenvalues and eigenvectors.

How to build any invertible linear transformation

Theorem

Any linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ with invertible matrix is obtained by composing reflections, stretches and shears.

Row Operations as Linear Transformations for 2x2 Matrices

Reflections

$$P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Stretching

$$M_1(k) = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, M_2(k) = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

Shearing

$$A_{21}(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, A_{12}(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

Eigenvalue and Eigenvector

Definition

Let λ be a scalar, x be a $n \times 1$ column vector and A be a $n \times n$ matrix. Any nontrivial vector that solves $Ax = \lambda x$ is called an **eigenvector**. If $Ax = \lambda x$ has a non-trivial solution, λ is an **eigenvalue**.

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Key idea: Eigenvectors are vectors sent to scalar copies of themselves under the linear map corresponding to A.

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For each eigenvalue λ, solve the linear system $(A - \lambda I_n)x = 0$ to find the eigenvectors.