Math 240: Solving Constant Coefficient Linear Differential Equations

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Outline

Today's Goals

1. Solving arbitrary order Constant Coefficient Linear Differential Equations.
2. Use the method of undetermined coefficients to solve nonhomogeneous Constant Coefficient Linear Differential Equations.

Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$$

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Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... a_1 \frac{dy}{dx} + a_0 y = 0,$$

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The Auxiliary Equation determines the general solution.

General Solution from the Auxiliary Equation

1. If m is a real root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.

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2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then
   $e^{\alpha x}cos(\beta x)$, $xe^{\alpha x}cos(\beta x)$, ... , $x^{k-1}e^{\alpha x}cos(\beta x)$ and
   $e^{\alpha x}sin(\beta x)$, $xe^{\alpha x}sin(\beta x)$, ... , $x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.

The Method of Undetermined Coefficients

Given a nonhomogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... a_1 y' + a_0 y = g(x)$$

where $a_n, a_{n-1}, ..., a_0$ are constants.

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2. Step 2: Find a particular solution by analyzing g(x) and making an educated guess.

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1. Step 1: Solve the associated homogeneous equation.
2. Step 2: Find a particular solution by analyzing g(x) and making an educated guess.
3. Step 3: Add the homogeneous solution and the particular solution together to get the general solution.

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e^{4x}	Ae^{4x}
$x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
$e^{2x} cos(4x)$	$Ae^{2x} sin(4x) + Be^{2x} cos(4x)$
$3xsin(5x)$	$(Ax + B)sin(5x) + (Cx + D)cos(5x)$
$xe^{2x} cos(3x)$	$(Ax + B)e^{2x} sin(3x) + (Cx + D)e^{2x} cos(3x)$

The Guessing Rule

The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiation of $g(x)$.

A Problem

Solve $y'' - 5y' + 4y = 8e^x$ using undetermined coefficients.

The solution

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by x^n, where n is the smallest positive integer that eliminates the duplication.