Math 240: Linear Differential Equations

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Outline

Today's Goals

Understand the form of solutions to the following types of higher order, linear differential equations

1. Initial Value Problems
2. Homogeneous and Nonhomogeneous Equations.

Solutions as a subspace

Theorem

(The Superposition Principle) The set of solutions to an nth-order homogeneous differential equation on an interval I form an n-dimensional vector subspace of $C^n(I)$. A basis for this space is called a **fundamental set**.

General Solutions to Nonhomogeneous Linear D.E.s

Theorem

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation on an interval I. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p$$

where the c_i are arbitrary constants.

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In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.

Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots m_1 and m_2, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

.

Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root m_1, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

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Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$$

.

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... a_1 \frac{dy}{dx} + a_0 y = 0,$$

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The Auxiliary Equation determines the general solution.

General Solution from the Auxiliary Equation

1. If m is a real root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.

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2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then $e^{\alpha x}cos(\beta x)$, $xe^{\alpha x}cos(\beta x)$, ... , $x^{k-1}e^{\alpha x}cos(\beta x)$ and $e^{\alpha x}sin(\beta x)$, $xe^{\alpha x}sin(\beta x)$, ... , $x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.