Math 240: Linear Differential Equations

Ryan Blair

University of Pennsylvania

Monday November 26, 2012

Outline

Today's Goals

Understand the form of solutions to the following types of higher order, linear differential equations

1. Initial Value Problems
2. Homogeneous and Nonhomogeneous Equations.

Differential equations

Definition

A differential equation is any equation involving a function, its derivatives.

Definition

A solution to a differential equation is any function that satisfies the equation.

A Few Famous Differential Equations

1. Einstein's field equation in general relativity
2. The Navier-Stokes equations in fluid dynamics
3. Verhulst equation - biological population growth
4. The Black-Scholes PDE - models financial markets

Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

$$Solve: \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

$$Solve: \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

an **nth-order initial value problem**(IVP) is the above equation together with the following constraint

$$Subject\ \ to: \quad y(x_0) = y_0, \ \ y'(x_0) = y_1, \ \ \ldots \ \ , y^{(n-1)}(x_0) = y_{n-1}$$

Higher Order Initial Value Problems

Definition

A **nth-order linear differential equation** is

$$Solve: \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ...a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

an **nth-order initial value problem**(IVP) is the above equation together with the following constraint

$$Subject \ \ to: \quad y(x_0) = y_0, \ \ y'(x_0) = y_1, \ \ \dots \ \ , y^{(n-1)}(x_0) = y_{n-1}$$

If $g(x) = 0$, then we say the differential equation is **homogeneous**.

Existence and Uniqueness

Theorem

Let $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.

Existence and Uniqueness

Theorem

Let $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial value problem exists on the interval and is unique.

Example: Does the following IVP have a unique solution? If so, on what intervals?

$xy''' + y'' - y' - cos(x)y = 9$ with $y(2) = 0$, $y'(2) = 0$ and $y''(2) = 0$

Solutions as a subspace

Theorem

(The Superposition Principle) The set of solutions to an nth-order homogeneous differential equation on an interval I form an n-dimensional vector subspace of $C^n(I)$. A basis for this space is called a **fundamental set**.

Example: Find the fundamental set for $x'' + x = 0$ using you intuition from calculus.

Review

Definition

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possess at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

General Solutions to Nonhomogeneous Linear D.E.s

Theorem

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation on an interval I. Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions to the associated homogeneous differential equation. Then the general solution to the nonhomogeneous equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p$$

where the c_i are arbitrary constants.

A Motivating Example

Our goal is to solve **constant coefficient** linear homogeneous differential equations.

A Motivating Example

Our goal is to solve **constant coefficient** linear homogeneous differential equations.

What if we guess $y = e^{mx}$ as a solution to $y'' + y' - 6y = 0$?

A Motivating Example

Our goal is to solve **constant coefficient** linear homogeneous differential equations.

What if we guess $y = e^{mx}$ as a solution to $y'' + y' - 6y = 0$?

What if we guess $y = e^{mx}$ as a solution to $ay'' + by' + cy = 0$?

A Motivating Example

Our goal is to solve **constant coefficient** linear homogeneous differential equations.

What if we guess $y = e^{mx}$ as a solution to $y'' + y' - 6y = 0$?

What if we guess $y = e^{mx}$ as a solution to $ay'' + by' + cy = 0$?

In this case, we get $e^{mx}(am^2 + bm + c) = 0$. There are three possibilities for the roots of a quadratic equation.

Case 1: Distinct Roots

If $am^2 + bm + c$ has distinct roots m_1 and m_2, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

.

Case 2: Repeated Roots

If $am^2 + bm + c$ has a repeated root m_1, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

.

Case 3: Complex Roots

If $am^2 + bm + c$ has complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution to $ay'' + by' + cy = 0$ is

$$y = c_1 e^{\alpha x} cos(\beta x) + c_2 e^{\alpha x} sin(\beta x)$$

.

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... a_1 \frac{dy}{dx} + a_0 y = 0,$$

the **Auxiliary Equation** is

$$a_n m^n + a_{n-1} m^{n-1} + ... a_1 m + a_0 = 0.$$

Auxiliary Equations

Given a linear homogeneous **constant-coefficient** differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 \frac{dy}{dx} + a_0 y = 0,$$

the **Auxiliary Equation** is

$$a_n m^n + a_{n-1} m^{n-1} + \ldots a_1 m + a_0 = 0.$$

The Auxiliary Equation determines the general solution.

General Solution from the Auxiliary Equation

1. If m is a real root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.

General Solution from the Auxiliary Equation

1. If m is a real root of the auxiliary equation of multiplicity k then e^{mx}, xe^{mx}, x^2e^{mx}, ... , $x^{k-1}e^{mx}$ are linearly independent solutions.
2. If $(\alpha + i\beta)$ and $(\alpha + i\beta)$ are a roots of the auxiliary equation of multiplicity k then
   $e^{\alpha x}cos(\beta x)$, $xe^{\alpha x}cos(\beta x)$, ... , $x^{k-1}e^{\alpha x}cos(\beta x)$ and
   $e^{\alpha x}sin(\beta x)$, $xe^{\alpha x}sin(\beta x)$, ... , $x^{k-1}e^{\alpha x}sin(\beta x)$ are linearly independent solutions.