Math 240: Linear Transformations of $\mathbb{R}^2$

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Friday November 2, 2012

Outline

Today's Goals

1. Know linear transformations of $\mathbb{R}^2$.

Matrices ARE linear transformations

Definition

A mapping $T : V \to W$ is a **linear transformation** if the following hold:

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
2. $T(cv) = cT(v)$ for all $v \in V$ and all scalars c.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. T is completely described by

$$T(v) = Av$$

where A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), ..., T(\mathbf{e}_n)]$$

and $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$.

Linear Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Reflections

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Stretching

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

Shearing

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

How to build any invertible linear transformation

Definition

If T and R are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrices A and B respectively, then their composition $T \circ R$ is a linear transforation with matrix AB.

Theorem

Any linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ with invertible matrix is obtained by composing reflections, stretches and shears.