Math 240: Systems of Differential Equations, Complex and Repeated Eigenvalues

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Outline

Today's Goals

1. Solve linear systems of differential equations with Complex Eigenvalues.
2. Solve linear systems of differential equations with non-diagonalizable coefficient matrices.

General Solution with Distinct Real Eigenvalues

Theorem

Let $A \in M_n(\mathbb{R})$. If A has n linearly independent eigenvectors $v_1, v_2, ..., v_n$, with **real** *eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ (not necessarily distinct), then the general solution to* **x'=Ax** *on any interval is*

$$\mathbf{X} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + ... + c_n v_n e^{\lambda_n t}$$

Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$$

Complex Eigenvalues

Theorem

Let $\lambda = a + bi$ *be a* **complex** *eigenvalue of* A *with eigenvectors* $v_1, ..., v_k$ *where* $v_j = r_j + is_j$. *Then the* $2k$ **real** *valued linearly independent solutions to* $x' = Ax$ *are:*

$$e^{at}(sin(bt)r_1 + cos(bt)s_1), ..., e^{at}(sin(bt)r_k + cos(bt)s_k)$$

and

$$e^{at}(cos(bt)r_1 - sin(bt)s_1), ..., e^{at}(cos(bt)r_k - sin(bt)s_k)$$

Repeated Eigenvalues

In a $n \times n$, constant-coefficient, linear system there are two possibilities for an eigenvalue λ of multiplicity 2.

1. λ has two linearly independent eigenvectors $\mathbf{K}_1$ and $\mathbf{K}_2$.
2. λ has a single eigenvector $\mathbf{K}$ associated to it.

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In the second case, there are linearly independent solutions $\mathbf{K} e^{\lambda t}$ and

$$[\mathbf{K} t e^{\lambda t} + \mathbf{P} e^{\lambda t}]$$

where we find $\mathbf{P}$ be solving $(\mathbf{A} - \lambda \mathbf{I})\mathbf{P} = \mathbf{K}$. $\mathbf{P}$ is a **generalized eigenvector**.

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How Bad Can it Get?

In general, you will only be asked to solve systems $X' = AX$ if the multiplicity of the eigenvalues of A is at most 1 more than the number of linearly independent eigenvectors for that value. In this case you need to find at most one vector $\mathbf{P}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{K}$

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Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$