Math 240: Systems of Differential Equations

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Outline

Today's Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Solve linear systems of differential equations using Eigenvalues.

If $A \in M_n(\mathbb{R})$

Given a **constant coefficient**, linear, homogeneous, first-order system

$$\mathbf{x'=Ax}$$

our intuition prompts us to guess a solution vector of the form

$$\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t}$$

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Hence, we can find such a solution vector iff K is an eigenvector for A with eigenvalue λ.

General Solution with Distinct Real Eigenvalues

Theorem

Let $A \in M_n(\mathbb{R})$. If A has n linearly independent eigenvectors $v_1, v_2, ..., v_n$, with **real** *eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ (not necessarily distinct), then the general solution to* **x'=Ax** *on any interval is*

$$\mathbf{X} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + ... + c_n v_n e^{\lambda_n t}$$

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Exercise: Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$

Complex Eigenvalues

Theorem

Let $\lambda = a + bi$ *be a* **complex** *eigenvalue of* A *with eigenvectors* $v_1, ..., v_k$ *where* $v_j = r_j + is_j$. *Then the* $2k$ **real** *valued linearly independent solutions to* $x' = Ax$ *are:*

$$e^{at}(cos(bt)r_1+sin(bt)s_1), e^{at}(cos(bt)r_2+sin(bt)s_2), ..., e^{at}(cos(bt)r_k+sin(bt)$$

and

$$e^{at}(cos(bt)r_1-sin(bt)s_1), e^{at}(cos(bt)r_2-sin(bt)s_2), ..., e^{at}(cos(bt)r_k-sin(bt)$$