# Math 240: Systems of Differential Equations

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# Outline

# Today's Goals

Combine linear algebra and differential equations to study systems of differential equations.

1. Define systems of differential equations
2. Develop the notion of Linear Independence.
3. Develop the notion of General Solution.

# Linear systems

## Definition

A **system of differential equations** is of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$$

Where $A(t)$ is an $n \times n$ matrix of functions, both $\mathbf{x}(t)$ and $\mathbf{b}(t)$ are $n \times 1$ matrices of functions and $\mathbf{x}'(t)$ is the $n \times 1$ matrix of derivatives of entries in $\mathbf{x}(t)$.
A **solution**

## Definition

Given a system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ a **solution vector** is an $n \times 1$ column matrix with differentialable functions as entries that satisfies the system.

## Definition

The following is an **initial value problem** for a first order system
$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ and $\mathbf{x}(t_0) = \mathbf{x}_0$

**Note**: As long as everything in sight is continuous on an interval $I$ containing $t_0$, then there exists a unique solution to the above IVP.

# Solutions to Linear Systems

Let $V_n(t)$ be the set of $n \times 1$ matrices with entries consisting of functions. $V_n(t)$ is a vector space under the natural operations of vector addition and scalar multiplication.

## Theorem

*Solutions to* **homogeneous** *systems of differential equations of the form*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

*form a vector subspace of* $V_n(t)$.

## Definition

Given an $n \times n$ homogeneous system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, a fundamental solution is a set of $n$ linearly independent solutions.

# The Wronskian

## Theorem

*Given an $n \times n$ homogeneous system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, the set of solutions to this system is an $n$ dimensional vector subspace of $V_n(t)$.*

## Theorem

*Let $X_1$, $X_2$, ..., $X_n$ be $n$ solution vectors to a homogeneous system on an interval $I$. If the* **Wronskian** *is non-zero at some $t_0$ in the interval $I$ then these vectors are linearly independent.*

# Guessing a Solution

Given a constant coefficient, linear, homogeneous, first-order system

$$\mathbf{x'=Ax}$$

our intuition prompts us to guess a solution vector of the form

$$\mathbf{x} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t}$$

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Hence, we can find such a solution vector iff $K$ is an eigenvector for $A$ with eigenvalue $\lambda$.

# General Solution with Distinct Real Eigenvalues

## Theorem

*Let* $\lambda_1$, $\lambda_2$, ..., $\lambda_n$ *be* $n$ **distinct** *real eigenvalues of the* $n \times n$ *coefficient matrix* **A** *of the homogeneous system* **X'=AX**, *and let* $\mathbf{K}_1$, $\mathbf{K}_2$, ... , $\mathbf{K}_n$ *be the corresponding eigenvectors. Then the general solution on* $(-\infty, \infty)$ *is*

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + ... + c_n\mathbf{K}_n e^{\lambda_n t}$$

*where the* $c_i$ *are arbitrary constants.*

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**Exercise:** Solve the linear system $X' = AX$ if

$$A = \begin{pmatrix} -1 & 2 \\ -7 & 8 \end{pmatrix}$$