1. Find the general solution to the following differential equation.

$$y^{(4)} + y''' + y'' = 0$$

Find the aux. Eq.

$$m^4 + m^3 + m^2 = 0$$

$$m^2(m^2+m+1) = 0$$

If $m^2=0$, $m=0$ if $m^2+m+1=0$ $m = \frac{-1 \pm \sqrt{1-4}}{2}$

$$m = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

Hence we have Linearly Independent solutions

$$e^{0x},\ x e^{0x},\ e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right),\ e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right)$$

The general solution is

$$Y = C_1 + C_2 X + C_3 e^{-\frac{1}{2}x}\cos\left(\tfrac{\sqrt{3}}{2}x\right) + C_4 e^{-\frac{1}{2}x}\sin\left(\tfrac{\sqrt{3}}{2}x\right).$$

2. Solve the following IVP given $y(0) = 1$ and $y'(0) = 0$.

$$y'' + 16y = 2cos(4x)$$

To find Y_h, find the aux. eq.

$$m^2 + 16 = 0$$

$$m = \pm 4i$$

So, $Y_h = C_1 \cos(4x) + C_2 \sin(4x)$.

To find Y_p, guess $Y_p = ax\cos(4x) + bx\sin(4x)$

$$Y_p' = -4ax\sin(4x) + a\cos(4x) + 4bx\cos(4x) + b\sin(4x)$$

$$Y_p'' = -16ax\cos(4x) - 4a\sin(4x) - 4a\sin(4x) - 16bx\sin(4x) + 4b\cos(4x) + 4b\cos(4x)$$

$$= -16ax\cos(4x) - 16bx\sin(4x) - 8a\sin(4x) + 8b\cos(4x)$$

Since $y'' + 16y = 2\cos(4x)$

$$-0ax\cos(4x) - 0bx\sin(4x) - 8a\sin(4x) + 8b\cos(4x) = 2\cos(4x)$$

$$-8a\sin(4x) + 8b\cos(4x) = 2\cos(4x)$$

$a = 0$ and $b = 1/4$

$$Y_p = 1/4\, x\sin(4x)$$

$$Y_g = Y_h + Y_p = C_1\cos(4x) + C_2\sin(4x) + 1/4\, x\sin(4x)$$

Since $y(0) = 1$

$$1 = C_1 + 0C_2 + 0$$

$$1 = C_1$$

Since $y'(0) = 0$

$$Y_g' = -4c_1 \sin(4x) + 4c_2 \cos(4x) + x\cos(4x) + \tfrac{1}{4}\sin(4x)$$

$$0 = -4c_1(0) + 4c_2(1) + 0 + \tfrac{1}{4}(0)$$

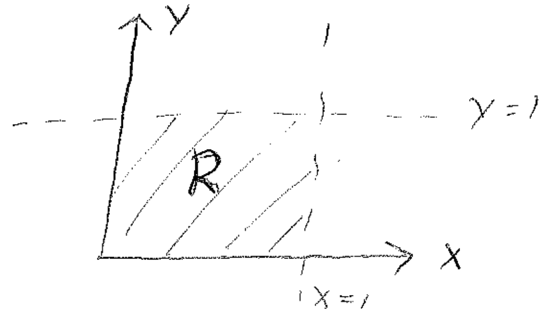
$$0 = c_2$$

Hence, the solution to the IVP is

$$\boxed{Y_g = \cos(4x) + \tfrac{1}{4}x\sin(4x).}$$

3. Evaluate the surface integral $\int\int_S G(x,y,z)dS$ given $G(x,y,z) = xy$ and S is the portion of the paraboloid $2z = 1 + x^2 + y^2$ in the first octant bounded by $0 \le x \le 1$ and $0 \le y \le 1$.

$$S: \; z = \tfrac{1}{2} + \frac{x^2}{2} + \frac{y^2}{2}$$



$$z_x = x$$
$$z_y = y$$

$$\iint_S G(x,y,z)\,dS = \iint_R G(x,y,z(x,y))\sqrt{1 + (z_x)^2 + (z_y)^2}\;dA$$

$$= \int_0^1 \int_0^1 xy\sqrt{1+x^2+y^2}\,dx\,dy$$

Let $u = 1+x^2+y^2$
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^1 \int_?^? y\sqrt{u}\,du\,dy$$

$$= \frac{1}{2}\int_0^1 \left. \frac{1}{2} y\,u^{-1/2}\right|_1^2 dy$$

$$= \frac{1}{2}\int_0^1 \left. \frac{1}{2} y\,(1+x^2+y^2)^{-1/2}\right|_0^1 dy$$

$$= \frac{1}{4}\int_0^1 y(2+y^2)^{-1/2} - y(1+y^2)^{-1/2}\,dy$$

$$= \frac{1}{4}\int_0^1 y(2+y^2)^{-1/2}\,dy + \frac{1}{4}\int_0^1 y(1+y^2)^{-1/2}\,dx$$

$$= \frac{1}{8}\int w^{-1/2}\,dw \qquad + \frac{1}{8}\int r^{-1/2}\,dr$$

$w = 2+y^2$
$dw = 2y\,dy$

$r = 1+y^2$
$dr = 2y\,dy$

$$= \frac{1}{2} \int_0^1 \frac{2}{3} y\, u^{3/2} \Big|_{?}^{?} \, dy$$

$$= \frac{1}{2} \int_0^1 \frac{2}{3} y \left(1+x^2+y^2\right)^{3/2} \Big|_0^1 \, dy$$

$$= \frac{1}{2} \int_0^1 \frac{2}{3} y \left(2+y^2\right)^{3/2} - \frac{2}{3} y \left(1+y^2\right)^{3/2} \, dy$$

$$= \frac{1}{3} \int_0^1 y \left(2+y^2\right)^{3/2} dy - \frac{1}{3} \int_0^1 y \left(1+y^2\right)^{3/2} dy$$

$$= \frac{2}{15} \cancel{\frac{1}{6}} \left(2+y^2\right)^{5/2} - \frac{2}{15} \left(1+y^2\right)^{5/2} \Big|_0^1$$

$$= \frac{2}{15} \left(3^{5/2} - 2^{5/2}\right) - \frac{2}{15} \left(2^{5/2} - 1\right)$$

$$= \boxed{\frac{2}{15} \left(3^{5/2} - 2^{7/2} + 1\right)}$$

4. Suppose there is a spring-mass system at rest at equilibrium with a mass of 2 kilograms attached to a spring with spring constant 32 $\frac{N}{m}$. Starting at time $t = 0$ a force of $f(t) = 68e^{-2t}cos(4t)$ is applied. Find the equation of motion in the absence of damping.

Let $x(t)$ be the eq. of motion

$$\frac{d^2x}{dt^2} = -\frac{\beta}{m}\frac{dx}{dy} - \frac{k}{m}x + \frac{f(t)}{m}$$

β = damping cont. = 0

k = spring cont. = 32

m = mass = 2

$f(t)$ = driving force = $68e^{-2t}\cos(4t)$

$$\boxed{\begin{array}{l} x(0) = 0 \\ x'(0) = 0 \end{array}}$$

$$x'' = -16x + 34e^{-2t}\cos(4t)$$

$$x'' + 16x = 34e^{-2t}\cos(4t)$$

Solve for homogeneous sol.

Aux Eq: $m^2 + 16 = 0$

$m = \pm 4i$

$$x_h = C_1\cos(4t) + C_2\sin(4t)$$

Solve for particular Sol.

Guess $x_p = Ae^{-2t}\cos(4t) + Be^{-2t}\sin(4t)$

$$x_p' = -4Ae^{-2t}\sin(4t) + (-2)Ae^{-2t}\cos(4t) + 4Be^{-2t}\cos(4t) + (-2)Be^{-2t}\sin(4t)$$

$$= (-4A - 2B)e^{-2t}\sin(4t) + (4B - 2A)e^{-2t}\cos(4t)$$

$$X_p'' = (-4A-2B)(4e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)) + (4B-2A)(-4e^{-2t}\sin(4t) - 2e^{-2t}\cos(4t))$$
$$= (-12A-16B)e^{-2t}\cos(4t) + (16A-12B)e^{-2t}\sin(4t)$$

Since $x'' + 16x = 34e^{-2t}\cos(4t)$

$$(-12A-16B)e^{-2t}\cos(4t) + (16A-12B)e^{-2t}\sin(4t) + 16(Ae^{-2t}\cos(4t) + Be^{-2t}\sin(4t)) = 34e^{-2t}\cos(4t)$$

$$(4A-16B)e^{-2t}\cos(4t) + (16A+4B)e^{-2t}\sin(4t) = 34e^{-2t}\cos(4t)$$

$$4A - 16B = 34$$
$$16A + 4B = 0 \Rightarrow B = \boxed{-4A}$$

$$4A + 64A = 34$$
$$68A = 34$$
$$\boxed{A = \frac{1}{2} \qquad B = -2}$$

$$X_p = \frac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)$$

$$\boxed{X = c_1\cos(4t) + c_2\sin(4t) + \frac{1}{2}e^{-2t}\cos(4t) - 2e^{-2t}\sin(4t)}$$

now sub. in $x(0)=0$ and $x'(0)=0$ to find c_1 and c_2.

5. Find the general solution to $(x+a)^2y'' + (x+a)y' - y = 0$. Note, your answer will incorporate the variable a.

Using our intuition from the Cauchy-Euler Eq,

Guess $(x+a)^m$ as a solution

$$y = (x+a)^m$$

$$y' = m(x+a)^{m-1}$$

$$y'' = m(m-1)(x+a)^{m-2}$$

$$(x+a)^2 m(m-1)(x+a)^{m-2} + (x+a)m(x+a)^{m-1} - (x+a)^m = 0$$

$$(x+a)^m\left(m(m-1) + m - 1\right) = 0$$

since $(x+a)^m \neq 0$ for all x, then

$$m(m-1) + m - 1 = 0$$

$$m^2 - m + m - 1 = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

$$\boxed{y = C_1(x+a) + C_2(x+a)^{-1}}$$

6. Find all $g(y)$ such that the following line integral is path independent.

$$\int_{(0,0,0)}^{(2,4,8)} [y^2x + g(y)]dx + [x^2y + x\cos(y)]dy + \ln((\cos(z))^2 + 1)dz = \int P\,dx + Q\,dy + R\,dz$$

The integral is Path-independent iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

Since $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0$, then only need

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

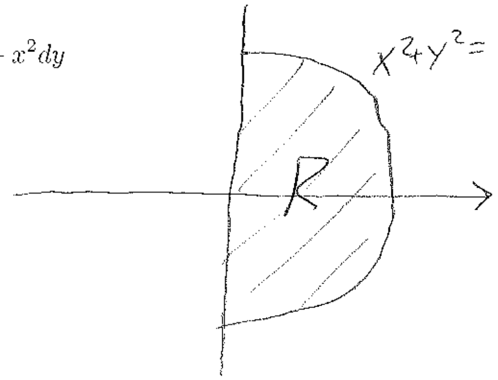
$$2xy + g'(y) = 2xy + \cos(y)$$

$$g'(y) = \cos(y)$$

$$\boxed{g(y) = \sin(y) + C}$$ where c is any constant

7. Evaluate the following line integral where C is the boundary of the region determined by the graphs of $x = 0$, $x^2 + y^2 = 1$ and $x \geq 0$

$$\oint_C [xy + ln(cos(e^{x^2} + 2))]dx + x^2 dy$$



By green's theorem

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} 2x - x \quad dx\, dy$$

$$= \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} x \quad dx\, dy$$

$$= \int_{-1}^{1} \frac{1}{2} (\sqrt{1-y^2})^2 \cancel{dx}\, dy$$

$$= \frac{1}{2} \int_{-1}^{1} 1 - y^2 \, dy$$

$$= \frac{1}{2} \left(y - \frac{1}{3} y^3 \right) \Big|_{-1}^{1}$$

$$= \frac{1}{2} \left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right)$$

$$= \frac{1}{2} \cdot \frac{4}{3} = \boxed{\frac{2}{3}}$$