MATH 240 - Spring 2011
Practice Midterm One

Name:

TA:

Recitation Time:

You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. Show all work, even on multiple choice or short answer questions—I will be grading as much on the basis of work shown as on the end result. Remember to put your name at the top of this page. Good luck.

Problem	Score (out of)
1	
2	
3	
4	
5	
6	
7	
Total	

1. List five properties an $n \times n$ matrix A can have that are equivalent to A being invertible.

1) $Ax = 0$ has a unique solution.

2) $\det(A) \neq 0$.

3) The rows of A are linearly independent.

4) The columns of A are linearly independent

5) RREF of A is I_n.

2. Find the determinant of the following matrix.

$$A = \begin{pmatrix} -1 & 4 & 0 & 0 & 1 \\ 0 & 3 & 0 & -11 & 0 \\ 2 & -3 & 1 & 4 & 3 \\ 0 & 0 & 0 & 5 & 0 \\ 2 & 19 & 2 & 3 & -1 \end{pmatrix}$$

$$\det(A) = (-1)^{4+4}(5)\begin{vmatrix} -1 & 4 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & -3 & 1 & 3 \\ 2 & 19 & 2 & -1 \end{vmatrix}$$

$$= 5\left((-1)^{2+2}(3)\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 2 & 2 & -1 \end{vmatrix}\right.$$

$$= 5 \cdot 3 \cdot \left((-1)\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 0 + (1)\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}\right)$$

$$= 5 \cdot 3 \cdot (-(-1-6) + (4-2))$$

$$= 15 \cdot (7+2)$$

$$= 135$$

3. Given the matrix A find the diagonal matrix D and the invertible matrix P such that $P^{-1}AP = D$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Find λ s.t. $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

Expand about 2nd row.

$$(1-\lambda)(-1)^{2+2}\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((1-\lambda)^2 - 1) = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$(1-\lambda)\lambda(\lambda - 2) = 0$$

$\lambda = 1$ or 0 or 2.

Find eigen vector for $\lambda = 0$.

Solve $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$-R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $\left.\begin{array}{l} x - y + z = 0 \\ \text{and} \\ y = 0 \end{array}\right\} \quad x = -z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \leftarrow \text{eigenvector}$$

Find the eigenvector for $\lambda = 1$.

Solve $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

So, $-y + z = 0$

$x \cancel{+} - y = 0$

Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

eigenvector

Find eigenvector for $\lambda = 2$.

Solve
$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$R_1 + R_3$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$-2R_2 + R_3$

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

eigenvector

So, $-x - y + z = 0$

and

$y = 0$

Hence, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. For what value of a does the following system have ∞-many solutions?

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Examine $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & a & 0 \end{array}\right]$

$-R_1 + R_2$ and $-R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 2 & a-1 & 0 \end{array}\right]$$

$\frac{1}{4} R_2$

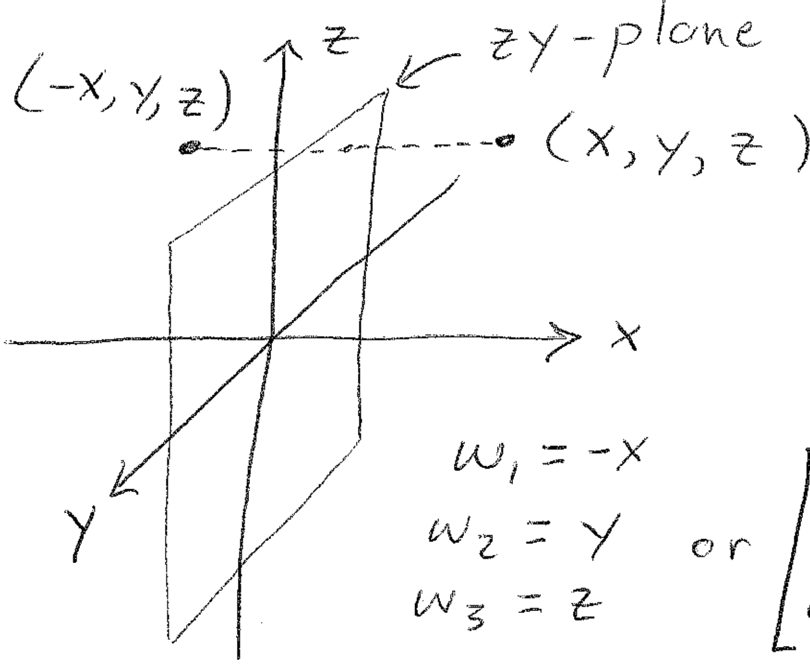
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 2 & a-1 & 0 \end{array}\right]$$

$-2R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & a-\frac{1}{2} & 0 \end{array}\right]$$

If $a - \frac{1}{2} = 0$, then the system has ∞-many solutions. Hence, $\boxed{a = \frac{1}{2}}$.

5. (10 points) Write the linear map in $\mathbb{R}^3$ given by reflection about the yz-plane in terms of matrix-vector multiplication. Find the eigenvalues and eigenvectors of this map.



$w_1 = -x$
$w_2 = y$
$w_3 = z$

or
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For this problem, use the geometry of the map to find eigenvalues & eigenvectors.

Eigenvalues $\lambda = -1$ or $\lambda = 1$.

Eigenvector for $\lambda = -1$ is $\langle 1, 0, 0 \rangle$.

Eigenvectors for $\lambda = 1$ are $\langle 0, 1, 0 \rangle$ and $\langle 0, 0, 1 \rangle$.

6. Solve the following system using matrix inverses.

$$x_1 + 2x_2 + 2x_3 = 1$$
$$x_1 - 2x_2 + 2x_3 = -3$$
$$3x_1 - x_2 + 5x_3 = 7$$

Examine $\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ 3 & -1 & 5 & 0 & 0 & 1 \end{array}\right]$

$-R_1 + R_2$ and $-3R_1 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 1 & 0 \\ 0 & -7 & -1 & -3 & 0 & 1 \end{array}\right]$$

$-\frac{1}{4}R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -7 & -1 & -3 & 0 & 1 \end{array}\right]$$

$7R_2 + R_3$ and $-2R_2 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & -1 & -\frac{5}{4} & -\frac{7}{4} & 1 \end{array}\right]$$

$2R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 1/4 & -1/4 & 0 \\ 0 & 0 & -1 & -5/4 & -7/4 & 1 \end{array}\right]$$

$(-1)R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 1/4 & -1/4 & 0 \\ 0 & 0 & 1 & -5/4 & -7/4 & 1 \end{array}\right]$$

Since $\begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$

then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 2 \\ 1/4 & -1/4 & 0 \\ -5/4 & -7/4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 1 \\ -11 \end{bmatrix}.$$

7. (Bonus!) Find the determinant of the following matrix.

$$A = (i + j)_{n \times n} \ for \ n \geq 3$$

$$A = \begin{bmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 3 & 4 & 5 & \cdots & n+2 \\ 5 & 6 & 7 & \cdots & n+3 \\ \vdots & \vdots & \vdots & & \vdots \\ n+1 & n+2 & n+3 & \cdots & n+n \end{bmatrix}$$

Examine $2(\text{Row } 2) - (\text{Row } 1)$

$= 2\langle 3, 4, 5, \ldots, n+2\rangle - \langle 2, 3, 4, \ldots, n+1\rangle$

$= \langle 6, 8, 10, \ldots, 2n+4) - \langle 2, 3, 4, \ldots, n+1\rangle$

$= \langle 4, 5, 6, \ldots, n+3\rangle$

$=$ Row 3

Hence, if $n \geq 3$ then the first 3 rows are linearly <u>dependent</u>.

By our theorem from clas, $\det(A) = 0$.