8. $\left(\frac{d}{dt}+7\right)\cdot\left(\frac{d}{dt}-1\right)x(t)=0 \rightsquigarrow x(t)=c_1 e^{-7t}+c_2 e^{t}$ for some constants c_1 & c_2

$x(0)=1 \Rightarrow c_1+c_2=1$

$|x(t)|\leq C$ for all $t\in\mathbb{R} \Rightarrow c_2=0 \Rightarrow c_1=1$, i.e. $\boxed{x(t)=e^{-7t}}$

9. $\left[\left(x\frac{d}{dx}\right)^2+x\frac{d}{dx}-1\right]u(x) = \left[\left(x\frac{d}{dx}+\frac{1}{2}\right)^2-\frac{5}{4}\right]u(x)$

$= \left(x\frac{d}{dx}+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x\frac{d}{dx}+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)u(x)$

Solution to the associated homogeneous equation is:

$c_1 x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}} + c_2\cdot x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}$

To get a special solution: try $a\cdot x^3$

$\left[\left(x\frac{d}{dx}\right)^2+x\frac{d}{dx}-1\right](ax^3) = a\cdot(9+3-1)x^3 = 11\cdot a\cdot x^3 \Rightarrow a=\frac{1}{11}$

i.e. general sol^n: $u(x)=\frac{1}{11}x^3+c_1 x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}+c_2 x^{-\frac{1}{2}-\frac{\sqrt{5}}{2}}$

$u(1)=5 \Rightarrow \frac{1}{11}+c_1+c_2=5$

$\lim\limits_{x\to 0^+} u(x)=0 \Rightarrow c_2=0$

$\Rightarrow c_1=\frac{54}{11}$, $\boxed{u(x)=\frac{1}{11}x^3+\frac{54}{11}x^{-\frac{1}{2}+\frac{\sqrt{5}}{2}}}$

10. Let $y(x)=x^2\cdot\log x$

$\left(x\frac{d}{dx}-2\right)x^2\log x = x^2 \Rightarrow \left(x\frac{d}{dx}-2\right)^2\left(x^2\log x\right)=\left(x\frac{d}{dx}-2\right)x^2=0$

i.e $y(x)$ satisfies $\boxed{x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=0}$

11. (a) $\iint_S \left[\operatorname{curl}\left(x^5z\,\vec{i}+xyz\,\vec{j}+e^{xz}\vec{k}\right)+z\vec{k}\right]\cdot\vec{N}\,dA$

$= \iint_S z\,\vec{k}\cdot\vec{N}\,dA = \iiint\limits_{\{x^2+y^2+z^2\leq 4\}} dx\,dy\,dz = \frac{4\pi}{3}\cdot 2^3 = \frac{32\pi}{3}$

(b) $\iint\limits_{S_r} \operatorname{curl}\left(x^5z\,\vec{i}+xyz\,\vec{j}+e^{xz}\vec{k}\right) = \int\limits_{\partial S_r}\left(x^5z\vec{i}+xyz\cdot\vec{j}+e^{xz}\vec{k}\right)\cdot d\vec{x}$

$= \int\limits_{\partial S_r} \vec{k}\cdot d\vec{x} = 0$

11. (Continued) $\iint_{S_r} z\vec{k}\cdot\vec{N}\,dA = \iint_{\{(x,y)\mid x^2+y^2\le 4,\ x\ge 0\}} \sqrt{4-x^2-y^2}\,dx\,dy + \iint_{\{(x,y)\mid x^2+y^2\le 4,\ x\ge 0\}} \sqrt{4-x^2-y^2}\,dx\,dy$

$$= \frac{1}{2}\iint_S z\,\vec{k}\,\vec{N}\,dA \underset{\text{by (a)}}{=} \frac{16\pi}{3}$$

So: $\iint_{S_r}\left[\operatorname{curl}\left(x^5z\,\vec{i} + xyz\,\vec{j} + e^{xz}\vec{k}\right) + z\vec{k}\right]\cdot\vec{N}\,dA = \frac{16\pi}{3}$

(c) $\iint_{S_r} y\vec{i}\,\vec{N}\,dA = \iint_{\{(y,z)\mid y^2+z^2\le 4\}} y\,dy\,dz = 0$

12. $ax + b\cos x + c\sin x = 0$

evaluate at $x=0 \Rightarrow b=0$

evaluate at $x=\frac{\pi}{2} \Rightarrow a\cdot\frac{\pi}{2} + c = 0$

evaluate at $x=\frac{5\pi}{2} \Rightarrow a\cdot\frac{5\pi}{2} + c = 0$

$\Big\} \Rightarrow a=c=0.$

13. (b) is true; (a), (c), (d), (e) are false

14. $\iint_S \vec{k}\cdot\vec{N}\,dA = \iint_S dx\,dy = \pi\cdot(2^2-1) = 3\pi$

15. The equation $\left[(x^3+ax^2+ax+1)\frac{d^2}{dx} + ax\cdot(x+1)\frac{d}{dx} + 1\right]u(x) = 0$

has an irregular singularity at $x = x_0$

$\Longleftrightarrow$ x_0 is a multiple root of the polynomial $f(x) := x^3+ax^2+ax+1$

and the multiplicity of x_0 as a root of $f(x)$ is 3 if $x_0 = 0$ or -1

Recall

x_0 is a multiple root

$\Longleftrightarrow$ x_0 is a root of $f(x)$ and is also a root of $f'(x)$

A calculation of this condition quickly leads to:

$-(a-3)^3\cdot(-2a^2+3a-3) = 0$. But the discriminant of the factor $-2a^2+3a-3$ is <0. So $f(x)$ has a multiple root only when $a=3$

15. (continued) From $x^3+3x^2+3x+1 = (x+1)^3$, we see that the differential equation has an irregular singularity at $x=-1$ when $a=3$. For other values of a the diffrential equation does not have any irregular singularity.

16. At the point $P=(-3,4,0)$, $\vec{N}(P) = \frac{1}{5}(-3\vec{i}+4\vec{j})$.

① First, compute the tangent line of C at P:

The tangent line is orthogonal to $-3\vec{i}+4\vec{j}$ and $\vec{i}+\vec{j}+\vec{k}$

$\Rightarrow \mathbb{R}\cdot\vec{t}(P) = \mathbb{R}\cdot(-4\vec{i}-3\vec{j}+7\vec{k})$

② Once we know $\vec{t}(P)$, we can determine the line $\mathbb{R}\cdot\vec{\nu}(P)$, because it is orthogonal to $\vec{N}(P)$ and $\mathbb{R}\cdot\vec{t}(P)$

$\Rightarrow \vec{\nu}(P) = \mathbb{R}\cdot\left(-4\vec{i}-3\vec{j}-\frac{25}{7}\vec{k}\right)$

③ $\vec{\nu}(P)$ points away from S at P

$$\Rightarrow \vec{\nu}(P) = \frac{1}{\sqrt{4^2+3^2+(\frac{25}{7})^2}}\cdot\left(-4\vec{i}-3\vec{j}-\frac{25}{7}\vec{k}\right)$$

because: $\left(-4\vec{i}-3\vec{j}-\frac{25}{7}\vec{k}\right)\cdot(\vec{i}+\vec{j}+\vec{k}) < 0$

$$\Rightarrow \vec{t}(P) = \vec{N}(P)\times\vec{\nu}(P) = \frac{1}{\sqrt{74}}\left(-4\vec{i}-3\vec{j}+7\vec{k}\right)$$

Let C_1 = the projection of C to the (x,y)-plane, with the orientation induced by the orientation of C

$\vec{t}(P) = (\text{a positive number})\cdot(-4\vec{i}-3\vec{j}+7\vec{k})$

$\Rightarrow C_1$ is oriented counterclockwise

Equation for C_1 is: $x^2+y^2+(1-x-y)^2=25$, i.e. $2x^2+2xy+2y^2-2x-2y=24$

$\Rightarrow (0,0)$ lies inside C_1

and $$\int_C \frac{-y\vec{i}+x\vec{j}}{x^2+y^2}\,d\vec{r} = \int_{C_1}\frac{-y\,dx+x\,dy}{x^2+y^2} = 2\pi$$