Math 240 practice problems for the final exam

April 30, 2011

1. Find the general solution to the following DE.

$$x^2y'' - xy' + 2y = 0$$

Guess $y = x^m$ to get

$$x^m(m(m-1) - m + 2) = 0$$

$$m^2 - 2m + 2 = 0$$

$$m = 1 \pm i$$

$$y_g = c_1 x^{1+i} + c_2 x^{1-i}$$

or

$$y_g = c_1 x \cos(\ln(x)) + c_2 x \sin(\ln(x))$$

2. Find the general solution to the following DE.

$$y'' - 2y' - 3y = \cos(x)$$

To find y_h use the Aux. Eq.

$$m^2 - 2m - 3 = 0$$

$$m = -1 \text{ or } 3$$

$$y_h = c_1 e^{-x} + c_2 e^{3x}$$

To find y_p guess $y_p = A\cos(x) + B\sin(x)$

$$-A\cos(x) - B\sin(x) - 2(-A\sin(x) + B\cos(x)) - 3(A\cos(x) + B(\sin(x))) = \cos(x)$$

After equating Coef.

$$B = \frac{-1}{10}$$

$$A = \frac{-1}{5}$$

$$y_g = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{5}\cos(x) - \frac{1}{10}\sin(x)$$

3. Evaluate the (unoriented) surface integral $\int\int_S G(x,y,z)dS$ given $G(x,y,z) = xz^3$ and S is the portion of the cone $z = \sqrt{x^2+y^2}$ inside the cylinder $x^2+y^2=1$.

$$\iint_S G\,dS = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\left(x^2+y^2\right)^{3/2}\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\;dx\,dy$$

Convert to polar

$$= \int_0^1\int_0^{2\pi} r\cos(\theta)\, r^{3/2}\sqrt{2}\; r\, d\theta\, d\cancel{\theta}r$$

$$= \int_0^1\int_0^{2\pi} \sqrt{2}\, r^{7/2}\cos\theta\, d\theta\, dr$$

$$= \int_0^1 0\, dr$$

$$= 0$$

4. (a) Find a number r such that the following differential equation

$$x^2y'' + (x-1)y' - y = 0$$

has a power series solution of the form

$$y(x) = x^r \cdot \left(1 + \sum_{n=1}^{\infty} a_n x^n\right).$$

(b) For the number r you gave in (a), give a formula for the coefficients a_n's which determines these coefficients recursively, and find a_1 and a_2.

In other words $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_0 = 1$.

After plugging-in

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty}(-1)a_n(n+r)x^{n+r-1} - \sum a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[a_n\left((n+r)(n+r)-1\right) - a_{n+1}(n+r+1)\right]x^{n+r} - a_0 r x^{r-1} = 0$$

Hence $r=0$ and $\boxed{a_n(n^2-1) = a_{n+1}(n+1)}$

$\boxed{a_0 = 1}$

$n=0$ $\quad a_1(1) = -a_0$

$\boxed{a_1 = -1}$

$n=1$ $\quad a_2(2) = a_1(0)$

$\boxed{a_2 = 0}$

$y(x) = 1 - x$

$n \geq 2$ $\quad a_n = 0$

5. Let B be the 3×3 matrix

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Find the characteristic polynomial $\det(T \cdot I_3 - B)$ of B.

(b) Is B diagonalizable? If so, find an invertible matrix C such that $C^{-1} \cdot B \cdot C$ is a diagonal matrix. If not, explain why such a matrix C does not exist.

(c) (extra) Find a formula for the powers B^n of B. Compute the exponential e^B of B and also the matrix-valued function e^{tB}.

a) $\det(B - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = -\lambda(\lambda - 1)^2$

b) B has only one eigenvector $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 1$, so B is <u>not</u> diagonalizable.

c) $B^n = \begin{bmatrix} 0 & 0 & 0 \\ -(n-2) & -(n-1) & -n \\ n-1 & n & n+1 \end{bmatrix}$ for $n \geq 2$

6. Let B be the 3×3 matrix

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Find the general solution of the system of linear ordinary differential equations

$$\frac{d}{dt}\vec{u}(t) = B \cdot \vec{u}(t), \quad \text{where } \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$

B has eigen val. 0 with eigenvec. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

and eigen val 1 with eigen vec $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

To find the generalized eigen vec.

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{general sol.} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{t} + c_3 \left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} te^{t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{t} \right)$$

7. Consider the matrix B with a parameter $a \in \mathbb{C}$,

$$B(a) = \begin{pmatrix} -a & a-1 \\ a & -a \end{pmatrix}.$$

Determine all values of the parameter a such that the matrix $B(a)$ is *not* diagonalizable.

Find $\det(B(a) - \lambda I) = \begin{vmatrix} -a-\lambda & a-1 \\ a & -a-\lambda \end{vmatrix} = \lambda^2 + 2a\lambda + a$

If $B(a)$ has distinct eigenvalues, then $B(a)$ is diagnolizable.

$\lambda^2 + 2a\lambda + a$ has a repealed root iff $4a^2 - 4a = 0$

or $a = 0, 1$.

When $a = 0$

The only eigenvalue of $B(a)$ is 0 and $B(a)$ has only a single eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence

$B(a)$ is not diagonalizable

When $a = 1$

The only eigenvalue of $B(a)$ is -1 and $B(a)$ has only a single eigen vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, $B(a)$ is not diagonalizable.