Math 240 practice problems for the final exam

April 30, 2011

1. Find the general solution to the following DE.

$$x^2y'' - xy' + 2y = 0$$

2. Find the general solution to the following DE.

$$y'' - 2y' - 3y = \cos(x)$$

3. Evaluate the (unoriented) surface integral $\int\int_S G(x,y,z)dS$ given $G(x,y,z) = xz^3$ and S is the portion of the cone $z = \sqrt{x^2+y^2}$ inside the cylinder $x^2+y^2=1$.

4. (a) Find a number r such that the following differential equation

$$x^2y'' + (x-1)y' - y = 0$$

has a power series solution of the form

$$y(x) = x^r \cdot \left(1 + \sum_{n=1}^{\infty} a_n\, x^n\right) .$$

(b) For the number r you gave in (a), give a formula for the coefficients a_n's which determines these coefficients recursively, and find a_1 and a_2.

5. Let B be the 3×3 matrix

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} .$$

(a) Find the characteristic polynomial $\det(T \cdot \mathrm{I}_3 - B)$ of B.

(b) Is B diagonalizable? If so, find an invertible matrix C such that $C^{-1}\cdot B\cdot C$ is a diagonal matrix. If not, explain why such a matrix C does not exist.

(c) (extra) Find a formula for the powers B^n of B. Compute the exponential e^B of B and also the matrix-valued function e^{tB}.

6. Let B be the 3×3 matrix

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} .$$

Find the general solution of the system of linear ordinary differential equations

$$\frac{d}{dt}\vec{u}(t) = B \cdot \vec{u}(t), \quad \text{where } \vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$

7. Consider the matrix B with a parameter $a \in \mathbb{C}$,

$$B(a) = \begin{pmatrix} -a & a-1 \\ a & -a \end{pmatrix}.$$

Determine all values of the parameter a such that the matrix $B(a)$ is *not* diagonalizable.

8. Suppose that a function $x(t)$ on $\mathbb{R}$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + 6\,\frac{dx}{dt} - 7\,x(t) = 0.$$

In addition we have $x(0) = 1$ and there exists a constant $C > 0$ such that

$$|x(t)| \leq C \qquad \text{for all } t \in \mathbb{R}\,.$$

Determine this function $x(t)$.

9. Find all solutions of the differential equation

$$\left[x^2\,\frac{d^2}{dx^2} + 2x\,\frac{d}{dx} - 1\right]\,u(x) = x^3$$

such that $u(1) = 5$ and $\lim_{x\to 0^+} u(x) = 0$

10. Find a non-trivial homogeneous linear ordinary differential equation satisfied by the function $x^2 \cdot \log x$.

11. Let S be the surface

$$S = \{(x, y, z \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4$$

oriented by the unit normal vector field

$$\vec{N}(x, y, z) := \frac{1}{2}(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}) \qquad \text{for all } (x, y, z) \in S\,.$$

Let S_r be the surface

$$S_r = \{(x, y, z \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 4,\ x \geq 0\}\,,$$

oriented by the restriction of $\vec{N}$ to S_r.

(a) Compute the oriented surface integral

$$\iint_S \left(\operatorname{curl}\left(x^5 z\,\vec{i} + xyz\,\vec{j} + e^{yz}\,\vec{k}\right) + z\,\vec{k}\right) \cdot \vec{N}\,\mathrm{d}A\,.$$

(This integral is also written as

$$\iint_S \left(\operatorname{curl}\left(x^5 z\,\vec{i} + xyz\,\vec{j} + e^{yz}\,\vec{k}\right) + z\,\vec{k}\right) \cdot \vec{N}\,\mathrm{d}S\,,$$

where $\mathrm{d}A$ is replaced by $\mathrm{d}S$.)

(b) Compute the oriented surface integral

$$\iint_{S_r} \left(\operatorname{curl}\left(x^5 z\,\vec{i} + xyz\,\vec{j} + e^{yz}\,\vec{k}\right) + z\,\vec{k}\right) \cdot \vec{N}\,\mathrm{d}A\,.$$

(c) Compute the oriented surface integral

$$\iint_{S_r} y\,\vec{i} \cdot \vec{N}\,\mathrm{d}A = \iint_{S_r} y\,\vec{i} \cdot \mathrm{d}S$$

12. Show that the functions $x, \cos x, \sin x$ are linearly independent. In other words, if a, b, c are real numbers such that

$$a\,x + b\,\cos x + c\,\sin x = 0 \qquad \text{for all } x \in \mathbb{R}\,,$$

then $a = b = c = 0$.

(Hint: Every value of x gives a linear relation between a, b and c.)

13. True/False questions. Let A be a 4×4 matrix with real entries such that $\det(T\cdot \mathrm{I}_4 - A) = (T-1)^2\,(T+1)^2$. For each of the following statements, determine whether it is true or false.

(a) A^2 is diagonalizable.

(b) If A^2 is diagonalizable then $A^2 = \mathrm{I}_4$.

(c) $A - \mathrm{I}_4$ is an invertible matrix.

(d) $A^2 - \mathrm{I}_4$ is an invertible matrix.

(e) $\dim\left(\{\vec{x} \in \mathbb{R}^4 \mid A^2 \cdot \vec{x} = \vec{x}\}\right) = 2$.

14. (extra) Let S be the surface obtained from the curve

$$C := \{(x, z) \in \mathbb{R}^2 \mid z = (x-2)(1-x),\ 1 \le x \le 2\}$$

on the (x, z)-plane by rotating C about the z-axis. In the cylindrical coordinates (r, θ, z), points on S satisfies

$$z = (r-2)(1-r), \quad 1 \le r \le 2\,.$$

Let $\vec{N}$ be the continuous unit normal vector field on S such that $\vec{N}(\frac{3}{2}, 0, \frac{1}{4}) = \vec{k}$. Orient the surface S by $\vec{N}$. Compute the oriented surface integral

$$\iint_S \vec{k} \cdot \vec{N}\, \mathrm{d}A \quad \left(= \iint_S \vec{k} \cdot \vec{N}\, \mathrm{d}S\right)$$

15. (extra) Consider the following ordinary differential equation with a parameter $a \in \mathbb{R}$,

$$\left[(x^3 + ax^2 + ax + 1)\,\frac{d^2}{dx^2} + ax(x+1)\frac{d}{dx} + 1\right] u(x) = 0\,.$$

For all but a finite number of real numbers a, the above differential equation has no irregular singularity at all points of $\mathbb{R}$, in the sense that for any $x_0 \in \mathbb{R}$ the differential equation is either ordinary at $x = x_0$ or has a regular singular point at $x = x_0$. Find the exceptional values of the parameter a for which the differential equation has a irregular singular point.

16. (extra) Let $S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 25,\ x + y + z \ge 1\}$, oriented by the unit normal vector field

$$\vec{N}(x, y, z) = \frac{1}{5}(x\vec{i} + y\vec{j} + z\vec{k}), \quad \text{for all } (x, y, z) \in S\,.$$

Let $C = \partial S$ be the boundary of S, the circle

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 25,\ x + y + z = 1\}\,.$$

Let $\vec{t}$ be the unit tangent vector field on C giving C the orientation so that Stokes theorem holds.

(a) Determine the value $\vec{t}(-3, 4, 0)$ of the vector field $\vec{t}$ at the point $(-3, 4, 0) \in C$.

(b) Compute the oriented line integral

$$\oint_C \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}\, \mathrm{d}\vec{r},$$

where C is oriented by the tangent vector field $\vec{t}$ on C.