Math 240: More Power Series Solutions to D.E.s at Singular Points

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Outline

Last Lecture!

Review of Last Time

1. Found power series solutions to D.E.s at regular singular points.

Given a differential equation $y'' + P(x)y' + Q(x)y = 0$

Definition

A point x_0 is an **ordinary point** if both $P(x)$ and $Q(x)$ are analytic at x_0. If a point in not ordinary it is a **singular point**.

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A point x_0 is a **regular singular point** if the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0. Otherwise x_0 is irregular.

Theorem

(Frobenius' Theorem)
If x_0 *is a regular singular point of* $y'' + P(x)y' + Q(x)y = 0$*, then there exists a solution of the form*

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r *is some constant to be determined and the power series converges on a non-empty open interval containing* x_0

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To solve $y'' + P(x)y' + Q(x)y = 0$ at a regular singular point x_0, substitute

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

and solve for r and the c_n to find a series solution centered at x_0.

Today's Goals

1. Deal with exceptional cases of finding power series solutions to D.E.s at regular singular points.

Indicial Roots

To find the r in $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ we substitute the series into $y'' + P(x)y' + Q(x)y = 0$ and equate the total coefficient of the lowest power of x to zero. This will be a quadratic equation in r.

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The roots, r_1 and r_2, we get are the **indicial roots** of $y'' + P(x)y' + Q(x)y = 0$

Cases

Case 1: If r_1 and r_2 are distinct and do not differ by an integer, then we get two linearly independent solutions

$$y_1 = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1} \quad and \quad y_2 = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$

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Case 2: In all other cases we get two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1} \ \ and \ \ y_2 = C y_1(x) ln(x) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$