Math 240: Power Series Solutions to D.E.s at Singular Points

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Outline

2 Lectures Left!

Review of Last Time

1. Found power series solutions to D.E.s at ordinary points.

Solving D.E.s Using Power Series

Given the differential equation $y'' + P(x)y' + Q(x)y = 0$, substitute

$$y = \sum_{n}^{\infty} c_n (x-a)^n$$

and solve for the c_n to find a power series solution centered at a.

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$$y'' - 2xy' + y = 0$$

Today's Goals

1. Find power series solutions to D.E.s at singular points.

Given a differential equation $y'' + P(x)y' + Q(x)y = 0$

Definition

A point x_0 is an **ordinary point** if both $P(x)$ and $Q(x)$ are analytic at x_0. If a point in not ordinary it is a **singular point**.

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A point x_0 is a **regular singular point** if the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0. Otherwise x_0 is irregular.

Theorem

(Frobenius' Theorem)
If x_0 is a regular singular point of $y'' + P(x)y' + Q(x)y = 0$, then there exists a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is some constant to be determined and the power series converges on a non-empty open interval containing x_0

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To solve $y'' + P(x)y' + Q(x)y = 0$ at a regular singular point x_0, substitute

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

and solve for r and the c_n to find a series solution centered at x_0.