Math 240: Vector Calc. Review

Ryan Blair

University of Pennsylvania

Thursday March 3, 2011

Outline

1. Review
2. Today's Goals
3. Vector-Valued Functions
4. Del, Div, Curl, Grad

Review for Last Time

1. Learned how to model spring/mass systems with damped motion.
2. Learned how to model spring/mass systems with driven motion.

Spring/Mass Example

Suppose 16 N of force stretches a spring 1 meter from equilibrium. If a mass of 4 kg is attached to the spring and subjected to a driving force given by $f(t) = cos(t)$, then find the equation that models the position of the system in the absence of a damping force.

Today's Goals

1. Review vector valued functions.
2. Review del, grad, curl and div.
3. Review line integrals.

Vector-Valued Functions

Definition

Vectors whose components are functions of a parameter t are called **vector-valued** functions.

$$r(t) =< f(t), g(t) >= f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$r(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}$$

Vector-Valued Functions

Definition

Vectors whose components are functions of a parameter t are called **vector-valued** functions.

$$r(t) =< f(t), g(t) >= f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$r(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}$$

Example: $r(t) =< cos(t), sin(t), t >$

Vector-Valued Functions

Definition

Vectors whose components are functions of a parameter t are called **vector-valued** functions.

$$r(t) =< f(t), g(t) >= f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$r(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{i}$$

Example: $r(t) =< cos(t), sin(t), t >$
Important: These are the parameterized curves we will integrate along

Derivative of a Vector-Valued Function

Definition

If $r(t) =< f(t), g(t), h(t) >$ where f, g, and h are differentiable, then

$$r'(t) =< f'(t), g'(t), h'(t) >$$

Derivative of a Vector-Valued Function

Definition

If $r(t) =< f(t), g(t), h(t) >$ where f, g, and h are differentiable, then

$$r'(t) =< f'(t), g'(t), h'(t) >$$

Theorem

(Chain Rule) If $\mathbf{r}$ *is a differentiable vector function and* $s = u(t)$ *is a differentiable scalar function, then*

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{s}}{ds}\frac{ds}{dt} = \mathbf{r}'(s)u'(t).$$

Integrating Vector-Valued Functions

Theorem

If f, g and h are integrable and $r(t) =< f(t), g(t), h(t) >$, then

$$\int \mathbf{r}(t)dt =< \int f(t)dt, \int g(t)dt, \int h(t)dt >$$

$$\int_a^b \mathbf{r}(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt >$$

Del and Grad

The differential operator **del** is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Del and Grad

The differential operator **del** is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Given a scalar function $f(x, y, z)$ we can form the **gradient of f** using del.

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Del and Grad

The differential operator **del** is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Given a scalar function $f(x, y, z)$ we can form the **gradient of f** using del.

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

∇f points in the direction of greatest change of f.

Del and Grad

The differential operator **del** is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Given a scalar function $f(x, y, z)$ we can form the **gradient of f** using del.

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

∇f points in the direction of greatest change of f.
Example: Guess the gradient of $f(x, y, z) = xyz$ at $(1, 1, 1)$ by interpreting the function as volume of a box.

Div and Curl

Definition

The **curl** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field

$$curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$$

Div and Curl

Definition

The **curl** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field

$$curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$$

Definition

The **divergence** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is given by the scalar function

$$div(F) = \nabla \dot{F} = \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial Q}{\partial y}\mathbf{j} + \frac{\partial R}{\partial z}\mathbf{k}$$

Div and Curl

Definition

The **curl** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the vector field

$$curl(F) = \nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$$

Definition

The **divergence** of a vector field $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is given by the scalar function

$$div(F) = \nabla \dot{F} = \frac{\partial P}{\partial x}\mathbf{i} + \frac{\partial Q}{\partial y}\mathbf{j} + \frac{\partial R}{\partial z}\mathbf{k}$$

Curl measures the tendency of a vector field to rotate. Divergence measures the tendency of a vector field to expand or contract.

Compositions

It is important to note

1. grad(scalar function) = vector field
2. div(vector field) = scalar function
3. curl(vector field) = vector field

Line Integrals in 2D

If G(x,y) is a scalar valued function and C is a smooth curve in the plane defined by the parametric equations $x = f(t)$ and $y = g(t)$ where $a \leq t \leq b$ then we can define the following line integrals

1. $\int_C G(x,y)dx = \int_a^b G(f(t), g(t))f'(t)dt$
2. $\int_C G(x,y)dy = \int_a^b G(f(t), g(t))g'(t)dt$
3. $\int_C G(x,y)ds = \int_a^b G(f(t), g(t))\sqrt{(f'(t))^2 + (g'(t))^2}dt$

Line Integrals in 3D

If G(x,y,z) is a scalar valued function and C is a smooth curve in 3-space defined by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $a \leq t \leq b$ then we can define the following line integrals

1. $\int_C G(x, y, z)dx = \int_a^b G(f(t), g(t), h(t))f'(t)dt$
2. $\int_C G(x, y, z)dy = \int_a^b G(f(t), g(t), h(t))g'(t)dt$
3. $\int_C G(x, y, z)dz = \int_a^b G(f(t), g(t), h(t))h'(t)dt$
4. $\int_C G(x, y, z)ds =$
   $\int_a^b G(f(t), g(t), h(t))\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}dt$