Math 240: Flux and Stokes' Theorem

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Outline

Review of Last Time

1. Learned how to find surface area.
2. Learned how to set up and evaluate surface integrals.
3. Learned what an orientable surface is.

Orientations

Definition

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Exercise Find the unit normal vectors to a sphere of radius a.

Today's Goals

1. Be able to set up and evaluate flux integrals.
2. Understand when and how to use Stokes' theorem.

Integrals of vector fields

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$$(F \circ \mathbf{n})\Delta S.$$

Where $\mathbf{n}$ is the normal vector to S and ΔS is the area of the patch of surface.

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The total volume per unit time is the **Flux** and is given by

$$\int\int_S (F \circ \mathbf{n}) dS$$

Example Let $T(x, y, z) = x^2 + y^2 + z^2$ model a temperature distribution in 3-space. From physics, heat flow is modeled by $F = -\nabla T$. Find the heat flow out of a sphere of radius a centered at the origin.

notation

Given a curve in 3-space C: $x = f(t)$, $y = g(t)$, $z = h(t)$. The position vector of C is $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.
$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = f'(t)dt\mathbf{i} + g'(t)dt\mathbf{j} + h'(t)dt\mathbf{k} = \mathbf{T}dt$
Where $\mathbf{T}$ is tangent vector to C.

Stokes' Theorem

Theorem

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve C. Let $F(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field for which P, Q and R are continuous and have continuous partial derivatives in the region of 3-space containing S. If C is traversed in the positive direction and $\mathbf{T}$ is the unit tangent vector to C then

$$\oint_C F \circ d\mathbf{r} = \oint_C (F \circ \mathbf{T})ds = \int\int_S (curl(F) \circ \mathbf{n})dS$$

where $\mathbf{n}$ is the unit normal to S in the direction of the orientation of S.

Example: Use Stokes' theorem to evaluate $\oint_C F \circ d\mathbf{r}$ where C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ oriented counterclockwise when viewed from above and $F = (2z + x)\mathbf{i} + (y - z)\mathbf{j} + (x + y)\mathbf{k}$.