Math 240: Double Integrals and Green's Theorem

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Outline

1. Review
2. Today's Goals
3. Double Integrals
4. Green's Theorem

Review of Last Time

1. Learned how to evaluate line integrals.
2. Learned how to test for path independence.

Theorem

(Fundamental theorem of Line integrals) Suppose there exists a function $\phi(x, y)$ *such that* $d\phi = P(x, y)dx + Q(x, y)dy$. *Then*

$$\int Pdx + Qdy = \phi(B) - \phi(A).$$

Test for path independence in 2D

Theorem

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C Pdx + Qdy$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x, y) in the region.

An Example

Question Evaluate the following integral and verify it is path independent.

$$\int_{(0,0)}^{(2,2)} (y^3 + 3x^2y)dx + (x^3 + 3y^2x + 1)dy$$

Today's Goals

1. Review how to evaluate double integrals in standard coordinates and polar coordinates.
2. Learn Green's Theorem and how to use it.

Intuition of line integrals in the plane

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Additionally, the sides of the object are vertical and the top of the object is the graph of the function $G(x,y)$.

Then the volume of the object is given by

$$\int\int_R G(x,y)dA$$

Where we are integrating with respect to the area of R.

Regions

Definition

A **Type I** region is given by the following formula

$$a \leq x \leq b, \;\; g_1(x) \leq y \leq g_2(x)$$

Definition

A **Type II** region is given by the following formula

$$c \leq y \leq d, \;\; h_1(x) \leq y \leq h_2(x)$$

Evaluation of Double Integrals

Theorem

Let f be continuous on a region R.
If R is Type I, then

$$\int\int_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

If R is Type II, then

$$\int\int_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example For the region R bounded by $y = x$, $x + y = 4$ and $x = 0$ evaluate

$$\int\int_R x + 1 dA$$

Example For the region R given by $0 \leq x \leq 2$, $x^2 \leq y \leq 4$ evaluate

$$\int\int_R xe^{y^2} dA$$

Evaluation of Double Integrals in Polar Coordinates

Theorem

Let f be continuous on a region R.
If R is Type PI, then

$$\int\int_R f(r,\theta)dA = \int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)rdrd\theta$$

If R is Type PII, then

$$\int\int_R f(r,\theta)dA = \int_{a}^{b}\int_{h_1(r)}^{h_2(r)} f(r,\theta)rd\theta dr$$

Change of Coordinates

If a region in the plane can be describe in polar coordinates as

$$0 \leq g_1(\theta) \leq r \leq g_2(\theta), \ \ \alpha \leq \theta \leq \beta$$

then wee have the following conversion formula

$$\int \int_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(rcos(\theta), rsin(\theta)) rdrd\theta$$

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Example Evaluate

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sqrt{x^2+y^2} dydx$$

Green's Theorem

Theorem (Green's Theorem)

Suppose C is a piecewise smooth simple closed curve bounding a region R. If P, Q, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous on R, then

$$\oint_C Pdx + Qdy = \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA,$$

where C is oriented counterclockwise.