Math 240: Line Integrals

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Outline

Review of Last Time

1. Reviewed vector valued functions.
2. Reviewed del, grad, curl and div.

An Example

Question Let $f(x, y, z) = zx - xy^2$. At the point $(1, 1, 1)$, find the angle between the vector pointing in the direction of fastest increase of $f(x, y, z)$ and the x-axis.

Today's Goals

1. Be able to parameterize curves in 2D and 3D.
2. Be able to evaluate line integrals.
3. Understand and apply the tests for path independence.
4. Use path independence to evaluate difficult line integrals.

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Then the total area of the fence is given by

$$\int_C G(x, y) ds$$

where we are integrating with respect to the arc length of C.

Line Integrals in 2D

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Line Integrals in 3D

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4. $\int_C G(x,y,z)dz = \int_a^b G(f(t), g(t), h(t))h'(t)dt$

Exact differentials and The Fundamental Theorem of Line Integrals

Definition

The **differential** of a function of two variables $\phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$P(x, y)dx + Q(x, y)dy$ is an **exact differential** if there exists a function $\phi(x, y)$ such that

$$d\phi = P(x, y)dx + Q(x, y)dy$$

The Fundamental Theorem of Line Integrals

Theorem

(Fundamental theorem of Line integrals) Suppose there exists a function $\phi(x, y)$ such that $d\phi = P(x, y)dx + Q(x, y)dy$. Then

$$\int Pdx + Qdy = \phi(B) - \phi(A).$$

Test for path independence in 2D

Theorem

Let P and Q have continuous first partial derivatives in an open simply connected region. Then $\int_C P dx + Q dy$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x, y) in the region.

Test for path independence in 3D

Theorem

Let P, Q and R have continuous first partial derivatives in an open simply connected region of space. Then $\int_C Pdx + Qdy + Rdz$ is independent of path C if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

for all (x, y, z) in the region.