Math 240: Spring-mass Systems

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Outline

Review for Last Time

1. Learned how to solve Cauchy-Euler Equations.
2. Learned how to model spring/mass systems with undamped motion.

Cauchy-Euler Equations

Definition

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is a **Cauchy-Euler equation**.

Note: These are **not** constant coefficient.

Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if m is a root of multiplicity k, then

$$x^m, \;\; x^m ln(x), \;\; \ldots \;\; , x^m (ln(x))^{k-1}$$

are k linearly independent solutions

Undamped Spring-Mass Systems

By Newton's Second Law and Hooke's Law, the following D.E. models an undamped mass-spring system

$$m\frac{d^2x}{dt^2} = -kx$$

where k is the spring constant, m is the mass placed at the end of the spring and $x(t)$ is the position of the mass at time t.

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Example: A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/sec. Find the equation of motion.

Today's Goals

1. Learn how to model spring/mass systems with damped motion.
2. Learn how to model spring/mass systems with driven motion.

Spring/Mass Systems with Damped Motion

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$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

is now our model, where m is the mass, k is the spring constant, β is the damping constant and $x(t)$ is the position of the mass at time t.

Changing Variables

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$$2\lambda = \frac{\beta}{m} \textit{ and } \omega^2 = \frac{k}{m}.$$

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and the roots of the Aux. Equation become

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2} \text{ and } m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

Case 1: Overdamped

If $\lambda^2 - \omega^2 > 0$ the system is **overdamped** since β is large when compared to k. In this case the solution is

$$x = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2-\omega^2}t} + c_2 e^{\sqrt{\lambda^2-\omega^2}t}).$$

Case 2: Critically Damped

If $\lambda^2 - \omega^2 = 0$ the system is **critically damped** since a slight decrease in the damping force would result in oscillatory motion. In this case the solution is

$$x = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3: Underdamped

If $\lambda^2 - \omega^2 < 0$ the system is **underdamped** since k is large when compared to β. In this case the solution is

$$x = e^{-\lambda t}(c_1 cos(\sqrt{\lambda^2 - \omega^2}t) + c_2 sin(\sqrt{\lambda^2 - \omega^2}t))$$

.

Example

A 4 foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers a damping force equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/sec.

Driven Motion

When an external force $f(t)$ acts on the mass on a spring, the equation for our model of motion becomes

$$\frac{d^2x}{dt^2} = -\frac{\beta}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{f(t)}{m}$$

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or in the language of λ and ω,

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$$

Example

When a mass of 2 kg is attached to a spring whose constant is 32 N/m, it comes to rest at equilibrium position. Starting at $t = 0$ a force of $f(t) = 65e^{-2t}$ is applied to the system. In the absence of damping, find the equation of motion.

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What is the amplitude of the oscillation after a very long time?