Math 240: Diagonalization and Eigenvalues

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Outline

Today's Goals

1. Have a deeper understanding of eigenvalues.
2. Be able to diagonalize matrices.
3. Be able to use diagonalization to compute high powers of matrices.

How to find Eigenvalues

To find eigenvalues we want to solve $Ax = \lambda x$ for λ.

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$Ax - \lambda x = 0$

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Hence, to find the eigenvalues, we solve the polynomial equation $det(A - \lambda I_n) = 0$ called the **characteristic equation**.

Rotation in the Plane

From last time, we saw the following matrix is rotation by angle θ about the origin in $\mathbb{R}^2$.

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

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Example: Show algebraically that rotation by $\frac{\pi}{4}$ has no eigenvalues

Important Examples

1. A matrix may have no eigenvaules (We don't count non-real eigenvalues)
2. A matrix may have multiple eigenvectors for a single eigenvalue.
3. A $n \times n$ matrix may not have n linearly independent eigenvectors.

Diagonalizability

Definition

An $n \times n$ matrix A is **diagonalizable** if there exists an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $P^{-1}AP = D$.

When A is diagnolizable, the columns of P are the eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

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Example: Find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonalizability Theorems

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Note: Not all diagonalizable matrices have n distinct eigenvalues.

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Example: Given

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

compute A^8.