Math 240: Diagonalization

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Outline

1. Review of Last Time

2. Diagonalizability

Review of last time

1. Interpret matrices as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.
2. Found eigenvalues.
3. Found eigenvectors.

How to find Eigenvalues

To find eigenvalues we want to solve $Ax = \lambda x$ for λ.

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$Ax - \lambda x = 0$

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Hence, we solve the polynomial equation $det(A - \lambda I_n) = 0$ called the **characteristic equation**.

Finding Eigenvectors

For each eigenvalue λ, solve the linear system $(A - \lambda I_n)x = 0$ to find the eigenvectors.

Today's Goals

1. Be able to diagonalize matrices.
2. Be able to use diagonalization to compute high powers of matrices.

Diagonalizability

Definition

An $n \times n$ matrix A is **diagonalizable** if there exists an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $P^{-1}AP = D$.

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Example: Verify that the following matrix is diagonalizable.

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

Diagonalizability Theorems

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A $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

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Note: Not all diagonalizable matrices have n distinct eigenvalues.