Math 240: Cauchy-Euler Equation

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Outline

1. Review
2. Today's Goals
3. Cauchy-Euler Equations
4. Spring-Mass Systems with Undamped Motion

Review for Last Time

1. Learned how to solve nonhomogeneous linear differential equations using the method of Undetermined Coefficients.

Solving a Nonhomogeneous Differential Equation

The general solution to a linear nonhomogeneous differential equation is

$$y_g = y_h + y_p$$

Where y_h is the solution to the corresponding homogeneous DE and y_p is any particular solution.

The Guessing Rule

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Example: Find the general solution to

$$y'' - y' = 4$$

The Fix to the Duplication Problem

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by x^n, where n is the smallest positive integer that eliminates the duplication.

Today's Goals

1. Learn how to solve Cauchy-Euler Equations.
2. Learn how to model spring/mass systems with undamped motion.

Cauchy-Euler Equations

Goal: To solve homogeneous DEs that are not constant-coefficient.

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Definition

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is a **Cauchy-Euler equation**.

The 2nd Order Case

Try to solve

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + ...cy = 0$$

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If m_1 and m_2 are distinct real roots to $am(m-1) + bm + c = 0$, then the general solution to this DE is

$$y = c_1x^{m_1} + c_2x^{m_2}$$

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Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if m is a root of multiplicity k, then

$$x^m, \;\; x^m ln(x), \;\; \ldots \;\; , x^m (ln(x))^{k-1}$$

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Example: What is the solution to

$$x^3 y''' + xy' - y = 0$$

Conjugate Complex Roots

Given the DE

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + ...cy = 0$$

If $am(m-1) + bm + c = 0$ has complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, then the general solution is

$$y_g = x^\alpha[c_1 cos(\beta ln(x)) + c_2 sin(\beta ln(x))]$$

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Example: Solve $25x^2y'' + 25xy' + y = 0$

Spring-Mass Systems with Undamped Motion

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Hooke's Law: The spring exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation.

$$F = ks$$

Note: k is called the spring constant

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Assuming free motion, **Newton's Second Law** states

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx$$

Solutions to Undamped Spring Equation

Question: What are the solutions to

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If $\omega^2 = \frac{k}{m}$ then the solutions are

$$x(t) = c_1 cos(\omega t) + c_2 sin(\omega t)$$

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