Math 240: Cauchy-Euler Equation

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Outline





- 3 **Cauchy-Euler Equations**
- 4 Spring-Mass Systems with Undamped Motion

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Review

Review for Last Time

Learned how to solve nonhomogeneous linear differential equations using the method of Undetermined Coefficients.

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Solving a Nonhomogeneous Differential Equation

The general solution to a linear nonhomogeneous differential equation is

$$y_g = y_h + y_p$$

Where y_h is the solution to the corresponding homogeneous DE and y_p is any particular solution.

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For a constant-coefficient nonhomogeneous linear DE, the form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiation of g(x).

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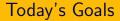
Example: Find the general solution to

$$y''-y'=4$$

Review

The Fix to the Duplication Problem

When the natural guess for a particular solution duplicates a homogeneous solution, multiply the guess by x^n , where *n* is the smallest positive integer that eliminates the duplication.



- Learn how to solve Cauchy-Euler Equations.
- Learn how to model spring/mass systems with undamped motion.

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Cauchy-Euler Equations

Goal: To solve homogeneous DEs that are not constant-coefficient.

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Cauchy-Euler Equations

Goal: To solve homogeneous DEs that are not constant-coefficient.

Definition

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

is a Cauchy-Euler equation.

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The 2nd Order Case

Try to solve

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + \dots cy = 0$$

by substituting $y = x^m$.

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The 2nd Order Case

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$$ax^{2}\frac{d^{2}y}{dx^{2}} + bx\frac{dy}{dx} + \dots cy = 0$$
$$y = x^{m}.$$

by substituting $y = x^m$.

If m_1 and m_2 are distinct real roots to am(m-1) + bm + c = 0, then the general solution to this DE is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

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Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if m is a root of multiplicity k, then

$$x^{m}, x^{m} \ln(x), \dots, x^{m} (\ln(x))^{k-1}$$

are k linearly independent solutions

Higher Order DEs and Repeated Roots

For a higher order homogeneous Cauchy-Euler Equation, if m is a root of multiplicity k, then

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are k linearly independent solutions

Example: What is the solution to

$$x^3y''' + xy' - y = 0$$

Conjugate Complex Roots

Given the DE

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + \dots cy = 0$$

If am(m-1) + bm + c = 0 has complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$, then the general solution is

$$y_g = x^{\alpha}[c_1 cos(\beta ln(x)) + c_2 sin(\beta ln(x))]$$

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Example: Solve $25x^2y'' + 25xy' + y = 0$

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Spring-Mass Systems with Undamped Motion

A flexible spring of length l is suspended vertically from a rigid support.

Spring-Mass Systems with Undamped Motion

A flexible spring of length l is suspended vertically from a rigid support.

A mass m is attached to its free end, the amount of stretch s depends on the mass.

Spring-Mass Systems with Undamped Motion

A flexible spring of length *I* is suspended vertically from a rigid support.

A mass m is attached to its free end, the amount of stretch s depends on the mass.

Hooke's Law: The spring exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation.

$$F = ks$$

Note: *k* is called the spring constant

Newton's Second Law

• The weight (W = mg) is balanced by the restoring force ks at the equilibrium position. mg = ks

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- If we displace from equilibrium by distance x the restoring force becomes k(x + s).

Assuming free motion, Newton's Second Law states

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx$$

Solutions to Undamped Spring Equation

Question: What are the solutions to

$$m\frac{d^2x}{dt^2} + kx = 0?$$

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Solutions to Undamped Spring Equation

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Question: What are the solutions to

$$m\frac{d^2x}{dt^2} + kx = 0?$$

If $\omega^2 = \frac{k}{m}$ then the solutions are

$$x(t) = c_1 cos(\omega t) + c_2 sin(\omega t)$$

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